

Dynamic Strategic Informed Trading with Risk-averse Market Makers

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Abstract

This paper presents an infinite horizon asset-pricing model to explain several financial anomalies. In our model, private information and public dividend (earnings) announcements follow a VAR process. There are risk-averse market makers who set prices competitively, a risk-averse informed trader whose trading incurs trade-size related costs and liquidity traders with possible overreaction (underreaction) to dividend announcements. The paper shows that trading cost and market impact cost lead to the short-term positive autocorrelation of stock returns (momentum), that the underreaction of liquidity traders to public news and the momentum trading of the informed trader lead to short-term post-earnings drift, and that the risk-aversion of the informed trader and the mean-reverting properties of private information contribute to the long-term negative autocorrelation of stock returns (reversal) and to long-term opposite post-earnings drift. In addition, our model shows that the variance of order flow contributes to the high volatility of stock returns relative to fundamental values. The model also has implications for market liquidity, trading volume, price volatility, and the incentives for traders to collect private information.

1 Introduction

Over the last two decades, persuasive empirical evidence has challenged the traditional view that security markets are efficient. The evidence calls into question whether security prices rationally incorporate public information. Some of the significant financial anomalies include: 1) the positive short-term autocorrelation of stock returns (momentum);¹ 2) the negative long-term autocorrelation of stock returns (reversal);² 3) after an earnings announcement, in the short-term, stock prices move in the same direction as the earnings surprise, but in the long term, prices move in the opposite direction of the earnings surprise (post-earnings announcement drift);³ and 4) prices are much more volatile relative to the present value of expected future dividends when the discount rate remains constant (excess return volatility).⁴

There is no consensus on the interpretation of this empirical evidence. There are two different approaches. This first approach is based on behavioral theory and assumes that investors are overconfident and that they self-attribute. For example, Daniel, Hirshleifer and Subrahmanyam (1998) explain these patterns by assuming that investors are overconfident about the precision of private information. Their overconfidence varies because of biased self-attribution, which means that when investors receive confirming public information, their confidence level increases, but when they receive disconfirming public information, their confidence level falls only modestly. Daniel et al. show that overconfidence implies negative long-run autocorrelation and excess volatility, and that biased self-attribution contributes to positive short-run autocorrelation (momentum) and short-run earnings drift.⁵

The other approach explains these anomalies using rational risk premiums. There are explanations for particular anomalies, however, we do not have an integrated risk premium theory to explain these phenomena. For example, Campbell and Kyle (1993) use noise shock to explain the “excess return volatility” puzzle, since risk-averse investors demand risk premiums for assuming any time-varying noise risk. Johnson (2002) considers a single-firm model with a standard price kernel. He shows that when expected dividend growth rates vary over time, his model generates momentum. Bansal and Yaron (2004) explain “excess return volatility” by modeling consumption and dividend growth rates as containing a small long-run predictable component and time-varying consumption volatility.

In this paper, we present an intertemporal equilibrium model that helps to explain these financial anomalies. We maintain the assumption that the stock market is semi-strong efficient, which means that prices incorporate public information. Our explanation is based on the time-

¹See Jegadeesh (1990), Jegadeesh and Titman (1993) and others.

²See Fama and French (1988), Poterba and Summers (1988) DeBondt and Thaler (1985, 1987) and others.

³See Ball and Brown (1968) and Bernard and Thomas (1989, 1990) and others.

⁴See Campbell and Shiller (1987), LeRoy and Parke (1992), Campbell and Kyle (1993) and others.

⁵See also Barberis, Shleifer and Vishny (1998) and Hong and Stein (1999).

varying liquidity premium that risk-averse market makers demand.⁶ We link this liquidity premium to order flow. Order flow is the key variable in market microstructure models, which is defined as the net orders from investors to market makers. The certain time-series properties of order flow contribute to the above puzzles.

This model includes several key ingredients. First, private information and public dividend announcements follow a VAR process, in which the dividend has a stochastic growth rate proportional to private information.⁷ Observing both private information and dividends is helpful for predicting the future values of dividends. Nevertheless, based on the history of dividends, the conditional expectation of private information is zero. Hence, current and future private information cannot be inferred from the history of public dividend information.

Second, there are three types of traders: the market maker, the liquidity trader and the informed trader. Market makers are risk-averse traders, who set prices competitively in response to both public news announcements and order flows from the liquidity traders and the informed trader. Since they observe public news, prices incorporate all public information. In addition, because they are risk averse, they demand liquidity premia for providing liquidity services to other traders. Such liquidity premia depend on the market makers' inventory positions and thus on the order flows from liquidity traders and the informed trader. When the net order from other traders is a sell (buy) order, the price is pushed down (up) to compensate for the higher (lower) risk that market makers must assume for their inventory.

Third, the net inventory of liquidity traders is exogenously given and follows a mean-reverting process with possible overreaction (underreaction) to public news surprises. Because liquidity traders are risk-averse, when they hold a long position they are reluctant to buy extra stocks and so they tend to sell. Similarly, when they hold a short position, they tend to buy. Hence, their positions are mean-reverting in the long run. These investors may have different opinions about a public news announcement.⁸ After good news, if overall they want to buy (sell), we say that they overreact (underreact) to such surprises.⁹

Fourth, there is a risk-averse informed trader who observes private information continuously and trades with quadratic trading costs. Because this informed trader trades on private information, if he buys (sells), the price is executed at a higher (lower) level on average.

⁶The market makers are liquidity providers who maintain the continuity of security markets. Because the market makers are risk-averse and the future is uncertain, they ask for liquidity premiums for holding stocks. If some investors want to buy, the market makers set prices to be higher than their estimated fundamental values since they hold a smaller inventory; if some investors want to sell, the market makers set prices lower than the estimated fundamental values since they hold a larger inventory. See Biais, Glosten and Spatt (2002) for a survey about risk-averse market makers.

⁷This bivariate representation of public and private information was first introduced in Campbell and Kyle (1993).

⁸See Harris and Raviv (1993) and Kendel and Pearson (1995).

⁹Rational investors know that risk-neutral market makers set prices incorporating public information. Therefore, they would not trade on this public news surprise. Underreaction and overreaction of liquidity traders to public news surprises are defined relative to this case.

Therefore, his trading moves the equilibrium price in the direction opposite to his desires and he incurs a so-called “market impact cost”. The trading cost and the market impact cost cause the informed trader to smooth his order over time. Hence, the order flow of the informed trader is positively autocorrelated in the short run. Since the informed trader is risk-averse, his position in stock is mean-reverting in the long run. Fifth, the risk-free interest rate is constant in this model. In Section 2, we briefly review the literature related to strategic informed trading to show our contributions regarding these features.

The above ingredients enable us to address these financial anomalies in an integrated way. When the market makers are risk averse, stock prices are the sum of three components: the present value of expected future dividends, the market makers’ estimate of future private information, and the liquidity premium demanded by risk-averse market makers. The liquidity premium is positively related to both the market makers’ estimations of the informed trader’s position and the liquidity traders’ position. Our explanation is based on the liquidity premium component of the price function. We find that when the market makers are risk-neutral, since they do not demand a liquidity (risk) premium for bearing risk, none of the financial anomalies mentioned above can exist in our model.

The key to understanding the momentum and reversal in stock returns is that order flows are positively autocorrelated in the short run and negatively autocorrelated in the long run. On the one hand, both the trading cost and market impact cost (“market friction”) cause the informed trader to smooth his order over time, which leads to short-term positive autocorrelation of the informed trader’s orders. Because the effect of this positive autocorrelation of orders from the informed trader dominates the negative autocorrelation of orders from the liquidity traders, order flow is positively autocorrelated in the short run. Because the stock return is positively related to order flow, stock returns are positively autocorrelated in the short run (momentum). On the other hand, because the informed trader is risk-averse, his position is mean-reverting in the long run. In addition, the position of liquidity traders and private information are also mean-reverting. Hence, those mean-reverting components in the stock price contribute to the observed long-term negative autocorrelation of stock returns (reversal).

When there is a positive earnings (dividend) announcement surprise, if liquidity traders underreact to this news, they sell some stock to the market makers. Since the market makers hold larger inventories in the stock, they set the price to be lower than their expectation of the long-term fundamental value. After this news announcement, on the one hand, since the informed trader knows the long-term value of the stock, he acts as an arbitrageur to buy stocks and adopts positive feedback trading (or momentum trading). At the same time, because

liquidity traders' positions are mean-reverting, they also buy back stock. Hence, the stock price is pushed up gradually in the short run (and accumulated excess return also increases). On the other hand, because private information is gradually incorporated into the price, the market makers' estimate of the long-term fundamental value decreases gradually to its true value.¹⁰ In addition, the position of the informed trader is also mean-reverting. He sells in the long run, which causes both the market makers inventory to increase and the liquidity premium to increase. Therefore, the price decreases in the long run and the accumulated excess return also decreases. The same logic works for a negative earnings announcement surprise.

Because the price is composed of the present value of expected future dividends based on the market makers' information set and the liquidity premium (which is related to order flow), price fluctuations depend on both the fluctuation of the fundamental value implied by a simple present value model of dividends with constant discount rate and on the fluctuations in order flows. Therefore, the fluctuation of order flows contributes to the "excess return volatility" puzzle.

This paper shows how to characterize a linear Bayesian Nash equilibrium. The characterization of the equilibrium falls into two parts. First, we show that given that market makers linearly update their beliefs about the primary state variables (private information I , the liquidity traders' net position U , and the informed trader's position X), the informed trader chooses his consumption and his order rate optimally. His order rate is a linear combination of the primary state variables I , U and X and the induced state variables \hat{I} , \hat{U} and \hat{X} , which are the market makers' estimates of these variables, respectively. Second, given that the informed trader chooses his order rate as a linear combination of his state variables, the stock price is a linear combination of the market makers' state variables. The market makers set stock prices competitively. They maximize their expected utility by choosing their consumption and their positions in the stock. In equilibrium, their positions clear the stock market. Using Separation Principle, the market makers' optimization problem can be decomposed into two steps. First, we show that they update their beliefs about the primary state variables in a linear manner. Second, we solve their optimization problem based on an equivalent representation of their information structure. We cannot have a closed-form solution. Hence, we develop a numerical procedure, termed the "recursive method", to deal with this problem. This method involves solving more than ten ordinary differential equations (ODEs) .

In addition to solving the above puzzles in an integrated framework, this model also has other implications for stock price, market liquidity, the properties of the informed trader's

¹⁰We will see why the market makers' estimate of the long-term fundamental value decreases in our model.

trading strategy and the incentive to collect private information.

One application of our model is in the study of the relationships among prices, the innovations in dividends, and order flows. When both private information and public dividend announcements arrive at the market continuously and the trade-size related cost is positive, we show that there exists a stationary linear equilibrium. Instantaneous price change depends on the entire path of order flows and the innovation in dividends. In contrast, in the simple present value model,¹¹ instantaneous price change depends only on the current innovation in dividends. In Kyle (1985), instantaneous price change depends only on current order flow.

This model is also helpful for understanding the positive relationships among the bid-ask spread, trading volume and price variability. The explanation advanced in Admati and Pfleiderer (1988) is that the clustering in noise trading leads to a clustering of informed trading, which generates the pattern found in empirical works. Our paper shows that a larger public news surprise (a higher instantaneous variance of dividends) can also generate this pattern. Intuitively, as the instantaneous variance of the dividends increases, private information becomes more valuable and the informed trader trades more aggressively. Trading volume is thus higher and the price also becomes more volatile. Because the market makers ask for more compensation due to asymmetric information between the informed trader and themselves, the portion of the bid-ask spread due to asymmetric information also increases. In this model, more liquidity trading leads to increases in trading volume and price variability. However, since the informed trader is risk-averse, he may trade less aggressively and the bid-ask spread can decrease (market depth increases).

Furthermore, we show that the trading of the informed trader is positively autocorrelated over a short period of time, but negatively autocorrelated over a longer period of time. Three factors determine this style of trading. The first factor is the existence of market impact costs and trading costs, which cause the informed trader to smooth his order over time and leads to a positive autocorrelation. The second factor is that both private information and the liquidity traders' inventory are mean-reverting, which contributes to the negative autocorrelation. The third factor is that the informed trader is risk-averse, which also contributes to the negative autocorrelation since he trades against his inventory of the stock. This increase in the risk aversion of the informed trader in particular leads to more negative autocorrelation at any horizon. The first factor dominates in the short run, while the latter two factors dominate in the long run.

The structure of the paper is as follows. We briefly review the related theoretical literature in Section 2. Section 3 specifies the notations and assumptions of the model. Section 4 obtains

¹¹The simple present value model means that the dividend follows a random walk and the discount rate (r) is constant.

the equilibrium by solving the market makers' learning problem and the informed trader's optimization problem. Section 5 addresses the momentum/reversal puzzle. Section 6 applies the impulse response function (IRF) to study the post-earnings announcement puzzle. Section 7 studies the excess return volatility puzzle. Section 8 contains an analysis of the equilibrium price. Section 9 investigates the properties of the informed trader's trading strategy and Section 10 concludes the paper. The proofs and figures are provided in several appendices.

2 Literature Review

In this section, we briefly review the theoretical research related to strategic informed trading, which falls into several categories.

The first category includes those papers that address the effects of the arrival of serial information. Admati and Pfleiderer (1988) assume that private information is short-lived. This paper explains the intra-day pattern in the volatility of security price and the volume of trading. Back and Pederson (1998) allow private information to change over the trading period. Private information is assumed to be long-lived. They show that the clustering of liquidity trading generates the same clustering of information use, and thus price volatility follows the same pattern as liquidity trading. Bernhardt, Seiler and Taub (2004) study informed trading in an infinite horizon model with multiple informed traders. These traders receive different private signals, which change over time. The focus of their paper is to develop a method to solve the stationary equilibrium in this scenario. However, in all these papers, public news announcements arrive at the market only at date zero. In addition, public news announcements and private information are assumed to be independent of each other. Hence, there is no interaction between innovations in public news and innovations in private information.

The second category of papers is to investigate the effects of the risk aversion of the informed trader. Subrahmanyam (1991) considers a one-shot Kyle-type model with risk-averse informed traders. He shows that there exists a linear equilibrium. However, it is hard to extend his model to a dynamic setup (multiple periods) and there is no linear solution in that case. Cho (2003) assumes that the informed trader is risk-averse, maintains the other assumptions in the context of Kyle (1985), and shows that market depth decreases over the trading period. However, this result may depend on the assumption of a finite horizon model. None of the above papers considers the case in which informed traders consume and observe new information during each period.

The third category examines the effect of differences in opinion about public information. Harris and Raviv (1993) assume that traders share common prior beliefs and receive common information but differ in the way they interpret this information. They show that their results

are consistent with the empirical findings about absolute price change and trading volume. Kandel and Pearson (1995) show that even without price changes, differences in opinion about public news can still generate trading. Cao and Ou-Yang (2003) develop a model of trading in stocks and options based on differences in opinion about public information among risk-averse investors. Their model shows that both additional trading sessions and the introduction of options enhance investors' perceived welfare and that in the presence of options, the trading volume of the underlying stock is positive even if the stock price remains unchanged. However, the above papers do not answer the question whether differences in opinion about public news affect market liquidity, informed trading or price dynamics.

The fourth category examines the effects of risk aversion of market makers. Subrahmanyam (1991) shows that when market makers are risk averse, the increased liquidity trading leads to reduced price efficiency.¹² However, there is no dynamic equilibrium model in which the informed trader trades strategically and market makers are risk averse. These market makers solve their optimization problems by choosing their consumption and their positions in stock.

Our paper is also closely related to Kyle and Genotte (1991). The differences between these two papers have four aspects. First, in Kyle and Genotte (1991), the liquidity traders are assumed to trade smoothly and their total inventory follows a deterministic process. Second, their paper does not consider the effects of the overreaction (underreaction) of liquidity traders to public news. Third, they assume that when the informed trader trades, he does not incur any trading costs. And fourth, they assume that the market makers are risk-neutral. The other assumptions are the same in these two papers.

3 The Model

We consider a continuous-time model over an infinite horizon $[0, \infty]$. Eight assumptions characterize our economy.

Assumption 1. There are two assets in this economy available for continuous trading: a risk-free bond and a risky stock. The bond yields a constant rate of return $1 + r$ ($r > 0$). Each share of stock generates a flow of dividends at an instantaneous rate D_t at time t .

Assumption 2. There are three types of traders in this economy: an informed trader, many liquidity traders, and competitive market makers.

Assumption 3. The private signal I is observed only by the informed trader. It follows an Ornstein-Uhlenbeck process:

$$dI_t = -\alpha_I I_t dt - \eta \sigma_D dB_1 + \sqrt{2\eta - \eta^2} \sigma_D dB_2, \quad (1)$$

¹²See also Biais, Glosten and Spatt (2002).

where α_I is a real positive constant and B_1 and B_2 are two standard independent Brownian motion processes. $-\alpha_I I$ gives the expected growth rate of private information. By Assumption 3, the steady-state level of private information is zero.¹³

Assumption 4. There are continuous dividend announcements to the public. The dividend rate D_t is governed by

$$dD_t = -\alpha_D(D_t - \bar{D})dt + \alpha_I I_t dt + \sigma_D dB_1, \quad (2)$$

where α_D is a real nonnegative number, $-\alpha_D(D_t - \bar{D})$ represents the expected growth rate of dividends, and \bar{D} is the steady-state level of the dividend rate D_t . Note that the expected growth rate of dividends at time t is positively related to I_t . The correlation coefficient between dD and dI is given by

$$\psi = -\frac{\eta\sigma_D^2}{\sqrt{2\eta}\sigma_D} = -\sqrt{\frac{\eta}{2}}, \quad (3)$$

which satisfies the following equation:

$$E\{I_{t+s}|D[-\infty, t]\} = 0, \quad (4)$$

where $s \geq 0$. Assumption 4 implies that private information is hidden in public dividend announcements. The econometrician cannot infer the private signal I_{t+s} from only the historical dividend process up to time t , although innovations in private information are negatively related to innovations in dividend rates. The private information process in this model lasts forever and will never be revealed to the public.

Assumption 5. For tractability, we do not explicitly model the trading behavior of liquidity traders. The demand of liquidity traders is exogenously given by

$$dU = -aUdt + \pi dB_1 + \sigma_U dB_3,$$

where a is a real positive constant and B_3 is a standard Brownian motion process, which is independent of B_1 and B_2 . The steady-state level of liquidity traders' net position is assumed to be zero. $\pi > 0$ implies that liquidity traders overreact to public dividend announcement surprises and $\pi < 0$ implies that liquidity traders under-react to public dividend announcement surprises.

(D, I, U) are the underlying processes in the economy. We write them as a Vector Auto-correlated (VAR) process:

$$\begin{pmatrix} dD \\ dI \\ dU \end{pmatrix} = \begin{pmatrix} -\alpha_D & \alpha_I & 0 \\ 0 & -\alpha_I & 0 \\ 0 & 0 & -a \end{pmatrix} \begin{pmatrix} D \\ I \\ U \end{pmatrix} dt + \begin{pmatrix} \sigma_D & 0 & 0 \\ -\eta\sigma_D & \sqrt{2\eta} - \eta^2\sigma_D & 0 \\ \pi & 0 & \sigma_U \end{pmatrix} \begin{pmatrix} dB_1 \\ dB_2 \\ dB_3 \end{pmatrix}. \quad (5)$$

¹³The half-life of this process is given by $\frac{\log 2}{\alpha_I}$, which measures the time for the expected value of I_t to reach the middle value between the current value I_t and the long run mean 0.

Assumption 6. There are N competitive risk-averse market makers, who can observe the history of order flows $\{d\omega_s, s \in (-\infty, t]\}$ in addition to public dividend announcements. They set the price competitively by maximizing their expected utility over an infinite horizon. They have exponential utility of the form:

$$U(C_{\{M,t\}}) = -\exp\left(-\rho t - \gamma_M C_{\{M,t\}}\right). \quad (6)$$

The order flow $d\omega$ and the innovation in dividend rate dD are the two variables observed by the market makers. The above assumptions imply that information asymmetry persists in this economy. Because the market makers cannot distinguish between the order flow from the informed trader and the net order flow from liquidity traders, the market makers can only partially infer private information from dividends, current order flow, and their historical values.

Assumption 7. The informed trader has monopoly power over private information. Thus, he will trade strategically by taking into account the impact of his trade on market price. He is infinitely lived and has exponential utility of the form:

$$U(C_t) = -\exp\left(-\rho t - \gamma C_t\right), \quad (7)$$

where ρ is the time-preference parameter, γ is his risk aversion coefficient, and C_t is the consumption rate at time t . He maximizes his expected utility over an infinite horizon. This CARA preference implies that the informed trader's demand is independent of his wealth. He chooses a consumption rate C_t and a trading strategy to maximize his expected utility over an infinite time horizon conditional on his information set. Following Kyle (1985) and Back (1992), we consider his trading strategy to be of the form $dX_t = \theta_t dt$.¹⁴ In addition, he is assumed to know the history of D , I and the market price P up to time t . Since he can observe price, he could infer liquidity traders' net inventory U . Hence, observing the price is equal to observing U . Thus, the information set of the informed trader at time t is $\mathcal{F}(t) = \{(D_s, I_s, U_s) : s \in [-\infty, t]\}$. Given that the informed trader's trading strategy is of the form: $dX = \theta dt$, we also assume that his trading incurs an instantaneous cost of $QC(\theta) = \frac{1}{2}k\theta^2 dt$ at time t .¹⁵

Assumption 8. The structure of the economy is common knowledge.

¹⁴In a competitive market, the investor chooses his position in stocks. See Merton (1969), Harris and Pliska (1981), Campbell and Kyle (1993) and Wang (1993). In this paper as well as in Kyle (1985) and Back (1992), the informed trader's inventory in stock X is a state variable and he chooses his order rate θ . Notice that X is differentiable with respect to t . The intuition is as follows. Suppose the price before trading is given by P_- . After the informed trader sends an order of dX , the price after trading is given by $P_- + dP$, because the market impact cost is given by $dPdX$, which is nonnegative. The informed trader tries to minimize this cost by smoothing his order over time. As a result, there is no martingale part in his order flow dX . We also exclude discrete orders by the informed trader (or there is a jump in his positions).

¹⁵In reality, trading costs mainly take the form of fixed and proportional costs. Here, the assumption of trade-size related cost is mainly for tractable reason. Trade-size related transaction cost induces the traders to split their orders. In the following parts, k refers to the trade-size related transaction cost.

4 The equilibrium

In this section, we solve for the equilibrium of the economy defined in the previous section. The equilibrium concept used in this paper is similar to the one developed by Kyle (1985). We consider only a linear equilibrium, which is decomposed into two parts. First, given the informed trader's trading strategy, the market makers set prices competitively by maximizing their expected utility. Since the market makers are competitive, it looks like that there is an auctioneer who sets the price for these market makers. The market makers maximize their expected utility by choosing their consumption and their positions in stock. Their positions clear the stock market and the stock price is a linear function of the market makers' state variables. Second, given the pricing rule and the market makers' updated beliefs, the informed trader's trading strategy is optimal, i.e, it maximizes his expected utility. The informed trader chooses his consumption and his order, which is a linear function of his state variables.

Before we solve for the equilibrium, let us discuss some unusual features of the equilibrium compared with competitive noisy expectation models.¹⁶ First, since the informed trader is risk-averse and smooths his order flow over time to minimize the impacts of his trading on market prices, his inventory position X is a state variable and is differentiable with respect to t . In contrast, in competitive noisy expectation models, the informed trader chooses his optimal position, which has a martingale part.¹⁷ Second, since the informed trader knows that the market makers update their beliefs about the state variables I , U and X conditional on the dividends and order flows, the informed trader's optimal order rate θ depends not only on the primary state variables I , U and X , but also on the induced state variables \hat{I} , \hat{U} and \hat{X} , which are the market makers' conditional estimates of I , U and X , respectively. Third, the market makers' signal (order flow) is affected by their beliefs about the primary state variables I , X and U , because these variables and the corresponding estimates of these variables all affect the informed trader's choice of his order rate θ .

4.1 The Price

In this economy, the uninformed market makers competitively set the price. Since they are risk-averse, they need some liquidity premium to compensate them for providing liquidity services. The following theorem shows that the price function in this model is composed of three parts: the present value of future dividends conditional on public dividend announcements, the present value of future private information conditional on their available information, and

¹⁶See Grossman (1976), Kreps (1977), Grossman and Stiglitz (1980), Hellwig (1980) and Wang (1993).

¹⁷For example, Wang (1993) considers a similar setup in which the price is competitively determined by both the informed and uninformed investors. In his model, their positions are linear combinations of the state variables, which are not differentiable with respect to time t .

a time-varying liquidity premium since the market makers bear an inventory risk.

Theorem 1. *For the economy defined by assumptions 1 - 9, if there exists a noisy rational expectation equilibrium, then the equilibrium price function is of the form*

$$P_t = P_{D,t} + \mu_1 \hat{I}_t + \mu_2 \hat{U}_t + \mu_3 \hat{X}_t, \quad (8)$$

where $P_{D,t} \equiv E[\int_0^\infty \exp(-rs) D_{t+s} ds | D[-\infty, t]] = \frac{\bar{D}}{r} + \frac{(D_t - \bar{D})}{(r + \alpha_D)}$ is the present value of future dividends based on public information. $\hat{I}_t \equiv E[I_t | \mathcal{F}_M(t)]$, $\hat{U}_t \equiv E[U_t | \mathcal{F}_M(t)]$, and $\hat{X}_t \equiv E[X_t | \mathcal{F}_M(t)]$ are the market makers' conditional expectations of I_t , U_t and X_t based on their information set at time t , respectively.

Remark 1. Some comments on this price function are in order. First, as we show later, \hat{I} , \hat{U} and \hat{X} and thus the current price P all depend on the entire path of dividends and order flows. Thus, the price is predictable. In contrast, Kyle (1985) shows that price depends only on the accumulated order flows. The simple present value model of expected future dividends, in which the dividend follows a random walk and the discount rate is constant, shows that the price depends only on the current dividend. Second, the price function reflects all public information and part of private information. In other words, the price is semi-strong form efficient.¹⁸ Third, $\mu_2 \hat{U} + \mu_3 \hat{X}$ reflects the compensations of the market makers for bearing risk. In this model, μ_1 , μ_2 and μ_3 are all positive constants. If either of \hat{U} or \hat{X} increases, then the market makers' inventory decreases. They bear less risk, therefore the price is pushed down. If the market makers are risk neutral, since they do not care about inventory risk, we can show that $\mu_2 = \mu_3 = 0$. In this case, since the market makers are competitive, the price is equal to the present value of expected future dividends conditional on the market makers' information set.¹⁹

4.2 The Kalman Filtering Problem of the Market Makers

The uninformed market makers learn about the value of I , U and X through their observation of the innovations in dividends dD and order flows $d\omega$. Since the order flows observed by them are affected by their beliefs about I , U and X , we need to conjecture the trading strategy of the informed trader to solve this filtering problem.

Although, in principal, the informed trader's optimal trading strategy at time t can depend on the history of all the primary variables he observes, since we are in a world of Markov chains, his optimal order rate depends only on the current values of a set of state variables. Here we conjecture that the sufficient set of state variables includes the current state variables I , U and X and the market makers' conditional expectation of these variables \hat{I} , \hat{U} and \hat{X} . In addition,

¹⁸See Roberts (1967).

¹⁹See Guo (2004) for more detail discussion of this case.

since the dividend is public information, the market makers' estimates of the dividend \widehat{D} is equal to D . Because the dividend is public information, D (or \widehat{D}) is not a state variable of the informed trader. The following Lemma shows that \widehat{X} is a linear combination of U , X and \widehat{U} . The order rate of informed trader thus depends only on I , U , \widehat{I} , \widehat{U} , and \widehat{X} . This lemma also reduces the effort required to solve the market makers' filtering problem.

Lemma 1. \widehat{U} and \widehat{X} satisfy the condition:

$$(U - \widehat{U}) = -(X - \widehat{X}). \quad (9)$$

Proof.

$$E[\omega_t | \mathcal{F}_M(t)] = \omega = X_t + U_t = \widehat{X}_t + \widehat{U}_t.$$

By rearrangement, we then have

$$(U_t - \widehat{U}_t) = -(X_t - \widehat{X}_t).$$

Q.E.D.

Assume that the market makers' conjecture is that the trading strategy of the informed trader is of the form:

$$\theta = f_1 I + f_2 U + g_1 \widehat{I} + g_2 \widehat{U} + g_3 \widehat{X}, \quad (10)$$

where θ is the order rate chosen by the informed trader. The optimal filter for I , U and X conditional on his observation of dividends and order flows, is derived in Appendix B.

There are some differences between the filtering problem considered in this paper and the classical Kalman filtering problem.²⁰ We summarize the result of the market makers' filtering problem in the following proposition.

Theorem 2. *Assuming that the market makers conjecture that $\theta = f^T Y = f_1 I + f_2 U + g_1 \widehat{I} + g_2 \widehat{U} + g_3 \widehat{X}$, the process of $(\widehat{I}, \widehat{U}, \widehat{X})^T$ is then given by*

$$\begin{pmatrix} d\widehat{I} \\ d\widehat{U} \\ d\widehat{X} \end{pmatrix} = CC \begin{pmatrix} \widehat{I} \\ \widehat{U} \\ \widehat{X} \end{pmatrix} dt + h d\omega + m [dD + \alpha_D (D - \bar{D}) dt]$$

²⁰See Chapter 13 of Hamilton (1994) for the discrete-time Kalman filter; see Lipster and Shiryaev (2001) for the continuous-time Kalman filter. Suppose that $\bar{Z}_t = E[Z_t | \mathcal{F}_M]$, where Z_t is an n -vector of state variables and S_t is an m -vector of signals. The filtering problem considered in this paper is given by

$$\begin{aligned} dZ_t &= [a_{z0} + a_{zz}Z_t + a_{zs}S_t + a_{\bar{z}z}\bar{Z}_t] dt + b_z dB_t, \\ dS_t &= [a_{s0} + a_{sz}Z_t + a_{ss}S_t + a_{\bar{z}s}\bar{Z}_t] dt + b_s dB_t, \end{aligned}$$

where B_t is a k -vector standard Wiener process. a_{z0} , a_{zz} , a_{zs} , $a_{\bar{z}z}$, a_{s0} , a_{sz} , a_{ss} , $a_{\bar{z}s}$, b_z , and b_s are $n \times 1$, $n \times n$, $n \times m$, $m \times 1$, $m \times n$, $m \times m$, $n \times k$, and $m \times k$ matrices of constants, respectively. In the classical Kalman filter problem, the drift of dZ_t and dS_t do not depend on \bar{Z}_t . Therefore, $a_{\bar{z}z}$ and $a_{\bar{z}s}$ are both zeros. Kyle and Genotte (1991) first mentioned this type of Kalman filtering problem.

$$\begin{aligned}
&= \begin{bmatrix} -\alpha_I(1+m_1) - h_1(f_1+g_1) & -h_1(f_2+g_2) + ah_1 & -g_3h_1 \\ -m_2\alpha_I - h_2(f_1+g_1) & -a(1-h_2) - h_2(f_2+g_2) & -g_3h_2 \\ -m_3\alpha_I + (1-h_3)(f_1+g_1) & ah_3 + (1-h_3)(f_2+g_2) & g_3(1-h_3) \end{bmatrix} \begin{pmatrix} \hat{I} \\ \hat{U} \\ \hat{X} \end{pmatrix} dt \\
&+ \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} d\omega + \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} [dD + \alpha_D(D - \bar{D})dt], \tag{11}
\end{aligned}$$

where $h = \begin{pmatrix} h_1 & h_2 & h_3 \end{pmatrix}^T$ and $m = \begin{pmatrix} m_1 & m_2 & m_3 \end{pmatrix}^T$ are 3×1 constants given by

$$\begin{aligned}
&\begin{bmatrix} (h_1\pi + m_1\sigma_D)\pi + h_1\sigma_U^2 & (h_1\pi + m_1\sigma_D)\sigma_D \\ (h_2\pi + m_2\sigma_D)\pi + h_2\sigma_U^2 & (h_2\pi + m_2\sigma_D)\sigma_D \\ (h_3\pi + m_3\sigma_D)\pi + h_3\sigma_U^2 & (h_3\pi + m_3\sigma_D)\sigma_D \end{bmatrix} \\
&= \Sigma \begin{pmatrix} f_1 & \alpha_I \\ f_2 - a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -\eta\sigma_D\pi & -\eta\sigma_D^2 \\ \pi^2 + \sigma_U^2 & \pi\sigma_D \\ 0 & 0 \end{pmatrix}. \tag{12}
\end{aligned}$$

The steady-state variance-covariance matrix $\Sigma_t = E \left[\begin{pmatrix} I - \hat{I} \\ U - \hat{U} \\ X - \hat{X} \end{pmatrix} \begin{pmatrix} I - \hat{I} & U - \hat{U} & X - \hat{X} \end{pmatrix} \right]$ is constant over time and is given by

$$\Sigma = M^T \bar{\Sigma} M, \tag{13}$$

where $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ and $\bar{\Sigma} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$ is given by

$$0 = a_0 \bar{\Sigma} + \bar{\Sigma} a_0^T + \bar{\Omega} - [\bar{\Sigma} \bar{q}_0^T + \bar{q}_1] (\bar{q}_s)^{-1} [\bar{\Sigma} \bar{q}_0^T + \bar{q}_1]^T, \tag{14}$$

where $\bar{\Omega} = \begin{pmatrix} 2\eta\sigma_D^2 & -\eta\pi\sigma_D \\ -\eta\pi\sigma_D & \pi^2 + \sigma_U^2 \end{pmatrix}$, $\bar{q}_0 = \begin{pmatrix} f_1 & \alpha_I \\ f_2 - a & 0 \end{pmatrix}$, $\bar{q}_1 = \begin{pmatrix} -\eta\sigma_D\pi & -\eta\sigma_D^2 \\ \pi^2 + \sigma_U^2 & \pi\sigma_D \end{pmatrix}$, $\bar{q}_s = \begin{pmatrix} \pi^2 + \sigma_U^2 & \pi\sigma_D \\ \pi\sigma_D & \sigma_D^2 \end{pmatrix}$, and $a_0 = \begin{pmatrix} -\alpha_I & 0 \\ 0 & -a \end{pmatrix}$. If $\alpha_I > 0$ and $a > 0$, then $\bar{\Sigma}$ can also be expressed as:

$$\begin{aligned}
\Sigma_{11} &= [2\eta\sigma_D^2 - h_1^2\sigma_U^2 - (h_1\pi + \sigma_D m_1)^2] / (2\alpha_I), \\
\Sigma_{12} &= -[\eta\pi\sigma_D + h_1 h_2 \sigma_U^2 + (\sigma_D m_1 + \pi h_1)(\sigma_D m_2 + \pi h_2)] / (a + \alpha_I), \\
\Sigma_{22} &= [\pi^2 + (1 - h_2^2)\sigma_U^2 - (h_2\pi + m_2\sigma_D)^2] / (2a). \tag{15}
\end{aligned}$$

Proof. See Appendix A.

Remark. Not observing I , U and X , the uninformed market makers rationally make inferences about I , U and X from order flows and public dividend information. They will partially attribute a shock in order flow to innovation in \hat{I} . Thus, $E[d\hat{I}d\omega] = h_1(\pi^2 + \sigma_U^2) + m_1\pi\sigma_D$. Similarly, we have $E[d\hat{U}d\omega] = h_2(\sigma_U^2 + \pi^2) + m_2\pi\sigma_D$. Although the market makers know that the informed trader trades “smoothly” ($dX = \theta dt$), they will still partially attribute a shock in order flow to innovation in \hat{X} . As a result, $E[d\hat{X}d\omega] = h_3(\sigma_U^2 + \pi^2) + m_3\pi\sigma_D$.

However, the market makers cannot perfectly infer I , U and X . We term h and m the update rules of the uninformed market makers with respect to order flows and public dividend announcements, respectively.

The following corollary shows that the information structure generated by $\mathcal{F}_M(t)$ has an equivalent representation generated by \bar{w}_M . This corollary helps us simplify the optimization problem of the market makers.

Corollary 1. \hat{I} , \hat{U} and \hat{X} also satisfy the following stochastic differential equations:

$$\begin{aligned} dy_c &= A_M y_c dt + C_M d\bar{w}_M, \\ &= (a_0 + b_0) y_c dt + [\Sigma q_0^T + q_{1zs}] q_{ss}^{\{-1/2\}} d\bar{w}_M, \end{aligned} \quad (16)$$

where Σ is defined in Theorem 2 and where $a_0 = \begin{pmatrix} -\alpha_I & 0 & 0 \\ 0 & -a & 0 \\ f_1 & f_2 & 0 \end{pmatrix}$, $b_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g_1 & g_2 & g_3 \end{pmatrix}$, $q_0 = \begin{pmatrix} f_1 & f_2 - a & 0 \\ \alpha_I & 0 & 0 \end{pmatrix}$, $q_{ss} = \begin{pmatrix} \pi^2 + \sigma_U^2 & \pi\sigma_D \\ \pi\sigma_D & \sigma_D^2 \end{pmatrix}$, $q_{1zs} = \begin{pmatrix} -\eta\sigma_D\pi & -\eta\sigma_D^2 \\ \pi^2 + \sigma_U^2 & \pi\sigma_D \\ 0 & 0 \end{pmatrix}$, and $d\bar{w}_M = q_{ss}^{\{-1/2\}} \left[\begin{pmatrix} d\omega - g^T y_c dt \\ dD + \alpha_D D dt \end{pmatrix} - \begin{pmatrix} f^T y_c - a\hat{U} \\ \alpha_I \hat{I} \end{pmatrix} dt \right]$. \bar{w}_M is a standard Wiener process with respect to $\mathcal{F}_M(t) = \mathcal{F}^{D,\omega}(t)$. In addition, the information structure generated by $\mathcal{F}^{\bar{w}}$ is equivalent to that generated by $\mathcal{F}_M(t) = \mathcal{F}^{D,\omega}(t)$.

The excess dollar return is given by

$$dQ = b_M y_c dt + d_M d\bar{w}_M,$$

where $b_M = \frac{\alpha_I}{\alpha_D + r} [1, 0, 0] + \mu^T A_M - r\mu^T$, $d_M = \mu^T C_M + [0, \frac{1}{\alpha_D + r}] q_{ss}^{\{1/2\}}$, and $\mu^T = (\mu_1, \mu_2, \mu_3)$.

4.3 Investment Opportunities

In this section, we solve the optimization problems of the informed trader and the market makers. To characterize the investment opportunities in this economy, let's define the instantaneous excess dollar return as $dQ = (D - rP)dt + dP$, which is the return of a zero-wealth portfolio that longs one share of stock financed fully by borrowing at the risk-free interest rate r . Given the process of I and U in Equation (5) and \hat{I} , \hat{U} and \hat{X} in Equation (11), the conjectured trading strategy $dX = [f_1 I + f_2 U + g_1 \hat{I} + g_2 \hat{U} + g_3 \hat{X}] dt$, and the price function in Theorem 1, it is easy to calculate the excess dollar return dQ and the stochastic process for the state variables I , U , \hat{I} , \hat{U} and \hat{X} . The result is shown in the following theorem.

Theorem 3. Let $Y^T = (I, U, \hat{I}, \hat{U}, \hat{X})$. Supposing that the market makers conjecture that $dX = \theta dt = f^T Y dt$, where $f^T = (f_1, f_2, g_1, g_2, g_3)$, then Y is of the following form:

$$dY \equiv A_Y Y dt + B_Y \theta dt + C_Y dB$$

$$\begin{aligned}
&= \begin{bmatrix} -\alpha_I & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 \\ 0 & 0 & -\alpha_I - h_1(f_1 + g_1) & -h_1(f_2 + g_2) & -g_3 h_1 \\ 0 & 0 & -h_2(f_1 + g_1) & -a - h_2(f_2 + g_2) & -g_3 h_2 \\ 0 & 0 & (1 - h_3)(f_1 + g_1) & (1 - h_3)(f_2 + g_2) & g_3(1 - h_3) \end{bmatrix} Y dt + \begin{bmatrix} 0 \\ 0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix} \theta dt \\
&+ \begin{bmatrix} -\eta \sigma_D & \sqrt{2\eta - \eta^2} \sigma_D & 0 \\ \pi & 0 & \sigma_U \\ h_1 \pi + m_1 \sigma_D & 0 & h_1 \sigma_U \\ h_2 \pi + m_2 \sigma_D & 0 & h_2 \sigma_U \\ h_3 \pi + m_3 \sigma_D & 0 & h_3 \sigma_U \end{bmatrix} dB, \tag{17}
\end{aligned}$$

where $d\omega = \theta dt + dU$, $dB = \begin{pmatrix} dB_1, & dB_2, & dB_3 \end{pmatrix}^T$, and $dX = \theta dt$. The excess dollar return is given by

$$dQ = dP + (D - rP)dt = q_0^T Y dt + \mu^T h \theta dt + q_1^T dB, \tag{18}$$

where q_0 and q_1 are functions of the market makers' update rule m, h and the informed trader's trading strategy f . They are given by

$$\begin{aligned}
q_0^T &= q_2^T + \mu^T (\alpha_I m, -ah, CC), \\
q_1^T &= \left[\frac{\sigma_D}{r + \alpha_D}, 0, 0 \right] + \mu^T (h [\pi, 0, \sigma_U] + \sigma_D m [1, 0, 0]),
\end{aligned}$$

where T denotes transpose. CC , h and m are defined in Theorem 2. $\mu^T = (\mu_1, \mu_2, \mu_3)$ and $q_2^T = (\frac{\alpha_I}{\alpha_D + r}, 0, r\mu_1, r\mu_2, r\mu_3)$.

Remark 1. This theorem is a direct result of Theorem 2, thus we omit the proof here. It shows how the trading of the informed trader affects the market makers' beliefs and the excess dollar return. $h\theta$ reflects how the trading of the informed trader influences the market makers' beliefs about I , U and X . $\mu^T h \theta$ reflects how the trading of the informed trader influences the expected excess dollar return, and $B_Y \theta$ reflects how the trading of the informed trader influences the expected value of the five state variables I , U , \hat{I} , \hat{U} and \hat{X} .

4.4 The Informed Trader's Optimization Problem

Let W denote the informed trader's wealth. We first derive the budget constraint of the informed trader. It is useful to consider the discrete-time counterpart. Suppose that he submits a market order $\theta_t = X_t - X_{t-1}$ and the price changes from P_{t-1} to P_t . The order is executed at price P_t . Execution of this market order $P_t(X_t - X_{t-1})$, interest payment ($r\bar{B}_{t-1}$), dividend payment ($D_t X_{t-1}$), trading cost ($\frac{1}{2} k \theta_t^2$) and consumption (C_t) cause his position in the risk-free bond to change by

$$\bar{B}_t - \bar{B}_{t-1} = r\bar{B}_{t-1} - P_t(X_t - X_{t-1}) + D_t X_{t-1} - C_t - \frac{1}{2} k \theta_t^2. \tag{19}$$

Hence, the change in his wealth is given by

$$\begin{aligned} W_t - RW_{t-1} &= (\bar{B}_t + P_t X_t) - R(\bar{B}_{t-1} + P_{t-1} X_{t-1}) \\ &= (P_t - rP_{t-1} + D_t)X_{t-1} - C_t - \frac{1}{2}k\theta_t^2. \end{aligned} \quad (20)$$

Extending this formula to continuous-time, the informed trader's budget constraint is then given by

$$dW = (rW - C - \frac{1}{2}k\theta^2)dt + XdQ, \quad (21)$$

where dQ is defined in Equation (18). It is similar to that of Merton (1969) and Harrison and Pliska (1981). The only restriction is that we consider a subset of the trading strategy considered by them.²¹

The informed trader's optimization problem is given by

$$\begin{aligned} \max_{\{\theta, C\}} E \left[- \int_{s=t}^{\infty} \exp(-\rho s - \gamma C_s) ds \right], \\ s.t. \quad dW = (rW - C - \frac{1}{2}k\theta^2)dt + XdQ. \end{aligned}$$

Let $J(W, Y, t)$ be the value function, where W and Y are the state variables governing the investment opportunities of the informed trader. It satisfies the Bellman equation:²²

$$\begin{aligned} 0 &= \max_{\{\theta, C\}} E [-\exp(-\rho s - \gamma C_s) + dJ(W, Y, t) | \mathcal{F}_t], \\ s.t. \quad dW &= (rW - C - \frac{1}{2}k\theta^2)dt + XdQ, \\ 0 &= \lim_{s \rightarrow \infty} E [J(W, Y, s) | \mathcal{F}_t]. \end{aligned} \quad (22)$$

Assuming that the market makers update their beliefs in the way given by Theorem 2, we conjecture that the value function of the informed trader is of the form:

$$J(W, Y, t) = -\exp \left[-\rho t - r\gamma(W + S_0) - \frac{1}{2}Y^T L Y \right], \quad (23)$$

where L is a 5×5 symmetric matrix and S_0 is a constant. The Bellman equation reduces to

$$\begin{aligned} 0 &= \max_{\{C, \theta\}} -\exp(-\rho s - \gamma C_s) + J_t + \frac{1}{2}tr(J_{YY}C_Y C_Y^T) + J_W(rW - C - \frac{1}{2}k\theta^2 + Xq_0^T Y) \\ &+ J_Y^T(A_Y Y + B_Y \theta) + \frac{1}{2}J_{WW}X^2 q_1^T q_1 + J_{YW}^T C_Y X q_1, \end{aligned}$$

where $J_W = -r\gamma J$, $J_t = -\rho J$, $J_Y = -LYJ$, $J_{WW} = (r\gamma)^2 J$, $J_{YY} = (-L + LYY^T L)J$, and $J_{YW} = r\gamma LYJ$. $tr(\cdot)$ denotes the trace of a matrix. q_0 , q_1 , A_Y , and C_Y are defined in Theorem 3. The solution to this problem is given in the following theorem.

²¹Note that this wealth process captures the wealth after the informed trader sends his order to the market makers.

²²To exclude the so-called doubling strategies of the informed trader, we require that $\lim_{s \rightarrow \infty} E[J(W, Y, t + s) | \mathcal{F}_t] = 0$, which is usually called the "transversality condition".

Theorem 4.1. *Given that $k > 0$ and the market makers conjecture that the informed trader's holding of stock is given by Equation (10), then Equation (22) has a solution given by*

$$J(W, Y, t) = -\frac{1}{r} \exp \left[-\rho t - Z_0 - r\gamma(W + \frac{V_0}{r}) - \frac{1}{2} Y^T L Y \right], \quad (24)$$

where L is a 5×5 symmetric matrix, $Z_0 = -(r - \rho)/\gamma$, and $V_0 = \frac{1}{2r\gamma} \text{tr}(C_Y^T L C_Y)$ represents the annualized steady-state value of his private information. His optimal order rate θ is given by

$$\theta = \frac{r\gamma\mu^T h X + Y^T L B_Y}{r\gamma k}. \quad (25)$$

His optimal consumption policy is given by

$$C = \frac{1}{\gamma} \left[r\gamma W + \gamma V_0 + (r - \rho)/r + 0.5 Y^T L Y \right]. \quad (26)$$

The value function parameter L satisfies the following Ricatti equation:

$$\begin{aligned} 0 &= rL - r\gamma(\bar{1}_3 q_0^T) - r\gamma(q_0 \bar{1}_3^T) - (A_Y^T L + L A_Y) \\ &+ L C_Y C_Y^T L - r\gamma k \left(\frac{L B_Y}{r\gamma k} + \frac{\mu h_1 \bar{1}_3}{k} \right) \left(\frac{L B_Y}{r\gamma k} + \frac{\mu h_1 \bar{1}_3}{k} \right)^T \\ &+ (r\gamma)^2 (q_1^T q_1) (\bar{1}_3 \bar{1}_3^T) + (r\gamma) L C_Y q_1 \bar{1}_3^T + (r\gamma) \bar{1}_3 q_1^T C_Y^T L, \end{aligned} \quad (27)$$

where $\bar{1}_3 = (0, -1, 0, 0, 1)^T$. q_0, q_1, A_Y and C_Y are defined in Theorem 3.

Proof. See Appendix B.

Remark 1. Equation (25) verifies that the conjectured form of the informed trader's trading strategy. In addition, the numerical results show that $f_1 + g_1 = 0$, that is, the informed trader trades only on his net private information.

4.5 The Market Makers' Optimization Problem

In this model, market makers set the price competitively, based on their information set. We consider a linear equilibrium in which the price is a linear combination of the market makers' state variables. Since they are competitive, the price is set as if there exists an auctioneer who sets the price and the market makers choose their optimal consumption and their optimal positions in the stock. Their positions clear the stock market. Since they are identical, we only need to consider one representative market maker's optimization problem.

Let $W_{\{i,M\}}$ be i th market makers's wealth, y_i is his holding of the stock and $C_{\{i,M\}}$ his consumption. His optimization problem is

$$\begin{aligned} \max_{y_i, C_{\{i,M\}}} & E \left[- \int_{s=t}^{\infty} \exp(-\rho s - \gamma_M C_{\{i,M,s\}}) ds \middle| \mathcal{F}_{\{M,t\}} \right], \\ \text{s.t.} & dW_{\{i,M\}} = (rW_M - C_{\{i,M\}}) dt + y_i dQ. \end{aligned} \quad (28)$$

The solution to this optimization problem is complicated, since the market makers' trading and consumption policies are functions of their information set, which depends on the whole history of dividends and order flows. However, given Corollary 1 of Theorem 2, we know that the information structure generated by $\mathcal{F}_M(t)$ has an equivalent representation generated by \bar{w}_M . Using this equivalent representation, we can solve the market makers' optimization problem as a standard Markovian one with the innovation process \bar{w}_M generating the dynamics. The market makers' state variables include their estimates of I , U and X . Let $Y_M = (\hat{I}, \hat{U}, \hat{X})^T$. Hence, the optimization problem of the market makers can be solved based on \bar{w}_M . In this case, the Separation Principle applies.²³

Corollary 1 in Theorem 2 gives the dynamics of Y_M and the excess dollar return based on $\mathcal{F}_M(t)$. Let $J_{\{M,i\}}(W_{\{M,i\}}, Y_M, t)$ be the value of function of the i th market maker. $J_{\{M,i\}}(W_{\{M,i\}}, Y_M, t)$ satisfies the following Bellman equation:

$$\begin{aligned} 0 &= \max_{\{y_i, C_{\{i,M\}}\}} E \left[-\exp(-\rho s - \gamma M C_{\{i,M\}}) + dJ_{\{M,i\}}(W_{\{M,i\}}, Y_M, t) | \mathcal{F}_{\{M,t\}} \right], \\ \text{s.t.} \quad dW_{\{i,M\}} &= (rW_{\{i,M\}} - C_{\{i,M\}})dt + y_i dQ, \\ 0 &= \lim_{s \rightarrow \infty} E \left[J_{\{M,i\}}(W_{\{M,i\}}, Y_M, t) | \mathcal{F}_{\{M,t\}} \right]. \end{aligned} \quad (29)$$

The solution to Equation (29) is given in Theorem 4.2. Since we consider only a symmetric solution, we omit subscript i .

Theorem 4.2. *Given that $k > 0$ and the market makers conjecture that the informed trader's holding of stock is given by Equation (10), then Equation (29) has a solution given by*

$$J(W_M, Y_M, t) = -\frac{1}{r} \exp \left[-\rho t - Z_{\{0,M\}} - r\gamma(W_M + \frac{V_{\{M,0\}}}{r}) - \frac{1}{2} Y_M^T L_M Y_M \right], \quad (30)$$

where L_M is a 3×3 symmetric matrix, $Z_{\{0,M\}} = -(r - \rho)/\gamma$, and $V_{\{M,0\}} = \frac{1}{2r\gamma} \text{tr}(C_M^T L_M C_M)$ represents the annualized steady-state value of his private information, where C_M is defined in Corollary 1 of Theorem 2. His optimal order rate θ is given by

$$y_M = f_M^T Y_M, \quad (31)$$

where $f_M = (f_{\{M,1\}}, f_{\{M,2\}}, f_{\{M,3\}})^T$ is (3×1) vector. His optimal consumption policy is given by

$$\bar{C}_M = \frac{1}{\gamma} \left[r\gamma W_M + \gamma V_{\{M,0\}} + (r - \rho)/r + 0.5 Y_M^T L_M Y_M \right]. \quad (32)$$

The value function parameter L_M satisfies the following Ricatti equation:

$$0 = rL_M + L_M C_M C_M^T L_M - L_M A_M - A_M^T L_M - (r\gamma)^2 d_M d_M^T f_M f_M^T. \quad (33)$$

Proof. See Appendix C.

²³For a detailed discussion of the Separation Principle, see Fleming and Rishel (1975).

4.6 Market Clearing

When the market clears, the sum of the net position of the liquidity traders, the position of the informed trader, and the net position of the market makers must be zero. We then have

$$U + X = \widehat{U} + \widehat{X} = -\sum_{i=1}^N (y_i).$$

Using Equation (31), we have the following equations:

$$\begin{aligned} f_{\{M,1\}} &= 0, \\ f_{\{M,2\}} &= -1/N, \\ f_{\{M,3\}} &= -1/N. \end{aligned} \tag{34}$$

4.7 The Numerical Procedure: The “Recursive Method”

To calculate the equilibrium, we need to solve a highly nonlinear system of equations, which include the market maker’s update rules h and m , the market makers’ trading strategy f_M , the market makers’ value function parameter L_M , the informed trader’s trading strategy f and the informed trader’s value function parameter L . We cannot have a closed-form solution and thus we resort to numerical procedures.

We develop a method to solve the problem numerically, which is termed the “recursive method” approach. Let us stack the endogenous parameters together and denote them LL . We first consider the corresponding problem over a finite horizon $[0, \bar{T}]$. In this case, all the endogenous parameters are functions of time t . We use backward induction to start from time \bar{T} and solve backwards differential equations to get the values of the endogenous parameters LL_t at t . We define the relative mean squared error $LMSE$ as:

$$\begin{aligned} LMSE^2 &\equiv ||f_n - f_{n-1}||^2 + ||\Sigma_n - \Sigma_{n-1}||^2 + ||CC_n - CC_{n-1}||^2 \\ &+ ||h_n - h_{n-1}||^2 + ||m_n - m_{n-1}||^2 + ||L_n - L_{n-1}||^2 + ||L_{\{M,n\}} - L_{\{M,n-1\}}||^2 \end{aligned} \tag{35}$$

We stop the above iteration once $LMSE$ is less than a given tolerance Tol , which is set to be 10^{-6} . In our numerical calculation, we also set $N = 2$.

5 Momentum and Reversal in Stock Returns

On the one hand, many empirical studies have shown that the return of both individual stock and index returns are positively autocorrelated (momentum) for holding periods less than 12 months.²⁴ Many people also think that such positive autocorrelation in individual stocks

²⁴See Lo and MacKinlay (1988), Jegadeesh (1990) and others.

is an important source of the momentum trading strategy.²⁵ On the other hand, a number of empirical studies have found significant a negative serial correlation in stock returns over a longer horizon of 3-5 years (reversal).²⁶ The negative autocorrelation of individual stock returns is considered to an important source of “contrarian” trading strategy. In this section, we employ our model to study the time series properties of stock returns over different holding periods.

For simplicity, we consider the return for the zero-wealth portfolio defined in the previous section. We can calculate the excess dollar return $Q(t) - Q(t - \tau)$ from Equation (18) and the expression of Y in Theorem 3. Q_t has the following solution (see e.g. Arnold (1974)):

$$Q_{t+\tau} - Q_t = \bar{k}^T(Y_{t+\tau} - Y_t) + v_{\{1,\tau\}}, \quad (36)$$

where \bar{k} satisfies $\bar{k}^T(A_Y + B_Y f^T) = q_2^T$ and $v_{\{1,\tau\}} = \int_t^{t+\tau} (q_1^T - \bar{k}C_Y)dB$.

The first-order serial correlation coefficient of $Q_{(t-\tau)} - Q_t$ is given by

$$\beta_Q(\tau) = \frac{Cov[Q_{(t-\tau)} - Q_t, Q_{(t+\tau)} - Q_t]}{Var[Q_{(t+\tau)} - Q_t]}, \quad (37)$$

where τ denotes the holding period.

When the market makers are risk neutral, since they are competitive, the price is equal to the present value of future dividends conditional on the market makers’ information set and the market makers’ expected profits are zero. As a result, the accumulated excess dollar return follows a martingale and the first order serial correlation $\beta_Q(\tau)$ is zero. When the market makers are risk-averse, they ask for compensation for bearing risk. If they have a larger (smaller) inventory (or $\hat{U} + \hat{X}$ is smaller (larger)), the stock price is pushed down (up) to compensate for future risk.

Because the informed trader trades strategically to minimize the impact of market impact costs and trading costs, he smooths his order over time. Hence, his orders are positively autocorrelated in the short run. If the positive autocorrelation of the orders from the informed trader dominates the negative autocorrelation of the orders from the liquidity traders, order flow is positively autocorrelated. The liquidity premium demanded by the market makers is given by $\mu_2\hat{U} + \mu_3\hat{X}$. Because the difference in the liquidity premium is positively related to order flow, the change in liquidity premium is also positively autocorrelated in the short run. As a result, stock returns are positively autocorrelated in the short run.

However, since the informed trader’s optimal order rate is negatively related to his inventory in stock, his position is mean-reverting in the long run. In addition, the position of the

²⁵See Jegadeesh and Titman (1993) and Daniel, Hirshleifer and Subrahmanyam (1998).

²⁶Fama and French (1985), Poterba and Summers (1988) and others.

liquidity traders and private information are both mean-reverting. These factors contribute to the long-term mean reversion of the price, thus the negative autocorrelation of stock returns.

Figure 1 plots $\beta_Q(\tau)$ against τ for different risk-averse coefficients of the market makers. When the market makers are risk neutral, $\beta_Q(\tau)$ is zero at any holding period τ . When the market makers are risk averse, this is not the case. For example, when $\gamma_M = 0.0015$, after about 3 months, as τ increases, $\beta_Q(\tau)$ decreases. $\beta_Q(\tau)$ is positive when the holding period is less than 1.2 year. When the holding period is larger than 1.2 year, $\beta_Q(\tau)$ is negative.

6 The Post-earnings Drift Puzzle

Many empirical studies have shown that after an earnings surprise, the price (or the excess return) moves in the direction of the surprise in less than three quarters, then moves in the opposite direction of the surprise. This phenomenon is called the “post-earnings drift puzzle”. In this section, we employ impulse response functions to study this issue.

We first discretize the continuous processes I , U , \hat{I} , \hat{I} and \hat{X} given in Theorem 3 by taking evenly spaced points $\{0, \Delta, 2\Delta, 3\Delta, \dots\}$. Similarly, we can also derive the dividend process. Given the above processes, we then use Equations (8) and (44) to calculate the price and the accumulated excess dollar return.

We assume that there is a positive dividend (earnings) announcement surprise in period 1, and the processes are at their steady state values: $I_0 = \hat{I}_0 = U_0 = \hat{U}_0 = X_0 = \hat{X}_0 = 0$ and $P_0 = 25$ in period 0. The magnitude of the shock is set to be 1. Since previous literature documented a unit root in the dividend process, we set $\alpha_D = 0$. We set $\alpha_I = 0.25$ and $a = 0.25$. We also set the real interest $r = 0.04$, $\gamma = 0.000005$, $\gamma_M = 0.001$, $\sigma_D = 0.5$, $\eta = 0.8$, $\pi = -100$, $k = 0.01$ and $\sigma_U = 4000$. We set $\Delta = 0.02$. Since there are 250 trading days in a year, the sample interval is a week.

After a positive dividend (earnings) announcement surprise, since private information is a stationary process, it decays over time. Since there is a unit root and a stationary component in the dividend, the level of the dividend converges to a constant. The market makers overestimate the level of private information and thus the estimation error $I - \hat{I}$ is negative. Since market makers observe better signals over time, the estimation error $I - \hat{I}$ increases monotonically to zero and the price converges to its true fundamental value over time. In the end, all private information is incorporated into price.

We first consider what happens in the short run after a positive dividend (earnings) announcement surprise. Initially, the liquidity traders underreact to the surprise and they sell some shares of their stocks. since the market makers bear more inventory risk, they set the price to be lower than their estimated long run fundamental value. If the amount of stock

sold by the liquidity traders is in a certain range, the price is lower than the true long-term fundamental value, although the stock price is still higher than its initial value. After the news announcement, because the informed trader knows the fundamental value, he buys in order to take advantage of arbitrage opportunities. In addition, because the position of the liquidity trader is mean-reverting, the liquidity traders also buy back what they sell initially. Hence the price increases in the short run after a positive dividends announcement surprise. Similarly, the accumulated excess dollar return $Q_t - Q_0$ also increases.

Since private information is gradually incorporated into the stock price, the market makers' estimate of the fundamental value decreases gradually to the true one. In addition, since the informed trader is risk averse, he then sells back the stock that he bought initially. Hence, the market makers demand a greater liquidity premium. These two factors play a dominant role in the long run. The stock price and the accumulated excess dollar return $Q_t - Q_0$ both decrease over a longer period of time after a positive dividend (earnings) announcement surprise.

Figure E plots the response of the economy to a positive dividend (earnings) announcement surprise in period 1. The pattern is consistent with the post-earnings drift puzzle.

7 Excess Price Variability

We measure price variability by the innovation variance of the price (i.e. the variance of the instantaneous return). Given the process of the equilibrium price in Proposition 1, we have

$$dp = \frac{1}{\alpha_D + r} dD + \mu_1 d\hat{I} + \mu_2 d\hat{U} + \mu_3 \hat{X}. \quad (38)$$

The innovation price variance is constant over time. The result is summarized in the following proposition.²⁷

Proposition 1. *Given assumptions 1 - 9, the innovation price variance is of the form*

$$\sigma_P^2 = \left[\frac{\sigma_D}{r + \alpha_D} + \mu^T h \pi + \mu^T m \sigma_D \right]^2 + \left[\mu^T h \sigma_U \right]^2,$$

where $\sigma_{P_D}^2 = \frac{\sigma_D^2}{(r + \alpha_D)^2}$ is the innovation variance of the present value of expected future dividends based on public dividend information and the variance of order flow: $\text{Var}(d\omega) = \left[(\mu^T h)^2 (\sigma_U^2 + \pi^2) \right]$.

Proof. See Appendix D.

The above proposition shows that the innovation price variance is related to both the innovation variance of public dividends and the innovation variance of order flows. Since the market makers are risk-averse, they ask for compensation for holding the stock. Hence, price

²⁷We also consider the effect of exogenous parameters on the unconditional price volatility $E[P_t^2]$. Since the results are similar, we omit them here.

variability deviates from the case in which the stock price is equal to the present value of expected dividends. Our model shows that the variance of order flows is useful in explaining the “excess volatility” of stocks prices.²⁸ If cross-section regressions of return volatility on the volatility of dividend surprises and the variance of order flows are performed, we would expect the coefficient of order flows to be positive.

8 Features of the Equilibrium Price

In this section, we study the properties of this equilibrium. One feature of the equilibrium price is its history dependence. The price dynamics given in Theorems 1 and 2 follow a complicated structure. Since the market makers do not have perfect information about the state of the economy, their estimates about I , X and U are based on past and present innovations in dividends and order flows. In contrast, in Kyle (1985), the accumulated order flow ω is a sufficient statistic of the current price P . There are several factors contributing to this history dependence. The first factor is that both private information and the net position of liquidity traders are mean-reverting processes.²⁹ The second factor is that the informed trader is risk averse, thus his order is negatively related to his inventory of the stock.

Kyle (1985) points out that market liquidity refers to several different elements of transaction costs, including “tightness”, “depth” and “resilience”. In this section, we focus on “market depth”. In the spirit of Kyle (1985), it is defined as the order flow necessary to induce the price to rise or fall by one dollar³⁰:

$$\lambda \equiv \frac{\partial dP}{\partial d\omega} = \mu^T h. \quad (39)$$

Note that market depth is constant over time in this model. In Kyle (1985), market depth is affected only by the variance of long-lived private information and the variance of noise trading. In our model, it is affected by many exogenous parameters.

8.1 The Positive Relationship among Trading Volume, Bid-ask Spreads, and Price Variability

There are many empirical papers documenting positive relationships among trading volume, the bid-ask spread and price variability. The classical explanation in Admati and Pfleiderer (1988) is that the clustering of noise trading leads to a clustering of informed trading, which generates the above patterns. However, the effects of public news announcements are not considered. When public dividends and private information are independent of each other,

²⁸See, Black (1986), DeLong et al. (1990), Campbell and Kyle (1993) and other papers.

²⁹Similarly, Andersen and Bollerslev (1997) show that the sum of several AR(1) signals can lead to the long-memory property of return volatility.

³⁰In the literature, it is usually called “Kyle lambda”.

then as the variance of public news surprises increases, the risk-averse informed trader trades less aggressively. Therefore, trading volume decreases and the bid-ask spread (due to asymmetric information) increases. We cannot see the above patterns. However, ample empirical evidence shows that when public news announcement surprises become more volatile, trading volume, the bid-ask spread and price variability all increase. In our model, because of the VAR process between private information and public news announcement surprises, trading volume, bid-ask spreads, and price variability are positively autocorrelated.

Since $dX = f^T Y dt$, we define the instantaneous trading volume rate as

$$TV = \sqrt{E[\theta^2 | \mathcal{F}_t]}.$$

Figure 4 plots the effects of the instantaneous variance of dividends on trading volume, the bid-ask spread and price variability when the market makers are risk-averse.³¹ As the instantaneous variance of dividends increases, since the unconditional variance of private information increases, the private information is more valuable to the informed trader. He trades more aggressively and trading volume increases. In addition, the market makers demand more compensation due to asymmetric information. As a result, trading volume, the bid-ask spread (which is the inverse of market depth) and price variability all increase.

9 The Optimal Investment Strategy of the Informed Trader

In this section, we examine the trading strategy of the informed trader. The optimal trading strategy of the informed trader is given by

$$dX = [f_1 I + f_2 U + g_1 \hat{I} + g_2 \hat{U} + g_3 \hat{X}] dt. \quad (40)$$

I reflects the private information observed by the informed trader. If $I > 0$ (< 0), the informed trader knows that future dividend payments tend to increase (decrease) and thus the stock is on average undervalued (overvalued); he then buys (sells). Hence, f_1 is a positive constant. Although we cannot prove it analytically, we find that in all our numerical results, $f_1 + g_1 = 0$. Therefore, the informed trader trades only on his net private information $I - \hat{I}$.

9.1 The Properties of the Informed Trader's Trading Strategy

We examine the time series properties of the informed trader's trading in this model economy. To do so, we discretize the process of $Y = (I, U, \hat{I}, \hat{U}, \hat{X})$ by sampling evenly.

³¹When the market makers are risk-neutral, this result still holds.

Theorem 3 implies that I , U , \hat{I} , \hat{U} and \hat{X} satisfy the discrete-time differential equation

$$\begin{pmatrix} I_{t+\tau} \\ U_{t+\tau} \\ \hat{I}_{t+\tau} \\ \hat{U}_{t+\tau} \\ \hat{X}_{t+\tau} \end{pmatrix} = e^{H\tau} \begin{pmatrix} I_t \\ U_t \\ \hat{I}_t \\ \hat{U}_t \\ \hat{X}_t \end{pmatrix} + u_{t+\tau}, \quad (41)$$

where $H = \begin{bmatrix} -\alpha_I & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 \\ 0 & 0 & -\alpha_I - h_1(f_1 + g_1) & -h_1(f_2 + g_2) & -g_3 h_1 \\ 0 & 0 & -h_2(f_1 + g_1) & -a - h_2(f_2 + g_2) & -g_3 h_2 \\ 0 & 0 & (1 - h_3)(f_1 + g_1) & (1 - h_3)(f_2 + g_2) & g_3(1 - h_3) \end{bmatrix}$, $u_{t+\tau} = \int_{s=0}^{\tau} e^{H(\tau-s)} F dB(t+s)$, and $F = \begin{bmatrix} -\sigma_D & \sqrt{2\eta - \eta^2}\sigma_D & 0 \\ \pi & 0 & \sigma_U \\ h_1\pi + m_1\sigma_D & 0 & h_1\sigma_U \\ h_2\pi + m_2\sigma_D & 0 & h_2\sigma_U \\ h_3\pi + m_3\sigma_D & 0 & h_3\sigma_U \end{bmatrix}$. At time t $O(t)$, the informed trader's order flow is defined as $X(t) - X(t - \tau)$. In order to look at the serial correlation in the informed trader's order flow, let us consider the following ratio:

$$\beta(\tau) = \frac{\text{Cov}[X_{(t+\tau)} - X_{(t)}, X_{(t)} - X_{(t-\tau)}]}{\text{Var}[X_{(t)} - X_{(t-\tau)}]}, \quad (42)$$

where τ denotes the length of the trading period. The following proposition shows how to calculate $\beta(\tau)$.

Proposition 2. Let $\Sigma_{\{\epsilon, \tau\}} \equiv \text{Var}(u_t)$ and $\Sigma_{\{Y, \tau\}} \equiv \text{Var}[X(t + \tau)]$. $\beta(\tau)$ is then given by

$$\beta(\tau) = \frac{\bar{1}_X^T \{A \Sigma_{\{Y, \tau\}} - A^2 \Sigma_{\{Y, \tau\}} - \Sigma_{\{Y, \tau\}}\} \bar{1}_X}{\bar{1}_X^T \{2 \Sigma_{\{Y, \tau\}} - A \Sigma_{\{Y, \tau\}} - \Sigma_{\{Y, \tau\}} A^T\} \bar{1}_X}, \quad (43)$$

where $\Sigma_{\{Y, \tau\}}$ satisfies $\Sigma_{\{Y, \tau\}} = A \Sigma_{\{Y, \tau\}} A^T + \Sigma_{\{\epsilon, \tau\}}$, $A = e^{H\tau}$, $\Sigma_{\{\epsilon, \tau\}} = \text{Var}(u_{t+\tau})$, and $\bar{1}_X = (0, -1, 0, 1, 1)^T$.

Proof. See Appendix E.

Figure 5 shows that the autocorrelation coefficient of the informed trader's order flow is positive over a short period of time when $\gamma_M = 0$. As the trading horizon τ increases, the autocorrelation coefficient decreases. Over a long period of time, the autocorrelation coefficient turns negative. Three factors determine this specific pattern. The first factor is the existence of market impact costs and trading costs, which cause the informed trader to smooth his order over time and leads to positive autocorrelation; the second factor is that private information and the liquidity traders' inventory are both mean-reverting, which contributes to negative autocorrelation. The third factor is that the informed trader is risk-averse, which also contributes to the negative autocorrelation since he trades against his inventory of stock.

In addition, this figure also shows when the informed trader is more risk-averse, since he also cares more about inventory risk; his order flow is therefore more negatively autocorrelated at any horizon. Figure 6 shows that when the quadratic cost k increases, his order flow is more positively autocorrelated at any horizon.

We can also calculate the excess dollar return $Q(t) - Q(t - \tau)$ from equation (18) and the expression for Y given in Theorem 3. Q_t has the following solution (see e.g. Arnold (1974)):

$$Q_{t+\tau} - Q_t = \bar{k}^T(Y_{t+\tau} - Y_t) + v_{\{1,\tau\}}, \quad (44)$$

where \bar{k} satisfies $\bar{k}^T(A_Y + B_Y f^T) = q_2^T$ and $v_{\{1,\tau\}} = \int_t^{t+\tau} (q_1^T - \bar{k}C_Y)dB$.

The cross-autocorrelation ratio between $X(t) - X(t - \tau)$ and $Q_{\{t+\tau\}} - Q_t$ is given by

$$\beta_P(\tau) = \frac{\text{Cov}[X_{(t-\tau)} - X_t, Q_{(t+\tau)} - Q_t]}{\sqrt{\text{Var}[X_t - X_{(t-\tau)}]} \sqrt{\text{Var}[Q_{(t+\tau)} - Q_t]}}. \quad (45)$$

Figure 7 shows that $\beta_P(\tau)$ is positive for any horizon. The implication is that the excess dollar return can be predicted from the past order flows of the informed trader.

10 Conclusion

This paper investigates a continuous-time equilibrium model to explain several well-documented financial anomalies. In our model, there are three types of traders: 1) a monopolistic risk-averse informed trader who trades strategically and incurs trade-size related costs; 2) liquidity traders whose net inventory is mean-reverting with potential overreaction (under-reaction) to public news; and 3) competitive risk-averse market makers. Public dividend announcements and private information follow a VAR process. We show that there exists a linear equilibrium and develop a numerical procedure to solve for the equilibrium.

The dynamic model considered in this paper is important for several reasons. First, it illustrates that the documented financial anomalies, including the momentum/reversal puzzle of stock returns, the post-earnings announcement drift puzzle, and the excess price volatility puzzle, are not inconsistent with a model in which prices incorporate public information and the liquidity providers and the informed trader are rational. These puzzles may be issues of market liquidity—liquidity providers ask for compensation for bearing risk. If market makers are risk-neutral, such anomalies do not exist in our model. Second, we demonstrate that the price has a long-memory property: it depends on the whole history of order flows and dividend announcement surprises. In addition, as public news becomes more intense, the bid-ask spread, the instantaneous return variance and trading volume all increase. We thus provide another explanation for the positive relationships among the bid-ask spread, return

volatility and the trading volume. Third, over a short period, the orders from the informed trader are positively autocorrelated since he trades smoothly over time. However, his orders are negatively autocorrelated over a longer period of time because the information process and the inventory of liquidity traders are both mean-reverting and the informed trader is risk-averse.

Appendix

A The Proof of Theorem 2

In this appendix, we solve the Kalman filter problem of the uninformed market makers.

Suppose that the optimal order rate of the informed trader is given by

$$\theta = f^T y + g^T y_c, \quad (46)$$

where $y = (I, U, X)^T$, $y_c = (\widehat{I}, \widehat{U}, \widehat{X})^T$, $f = (f_1, f_2, 0)^T$, and $g = (g_1, g_2, 0)^T$. Conjecture that y_c follows the following stochastic process:

$$dy_c = Cy_c dt + h d\omega + m(dD + \alpha_D D),$$

where $C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$, $h = (h_1, h_2, h_3)^T$, $m = (m_1, m_2, m_3)^T$, and T denotes transpose.³² The process of y can be written in the following form:

$$dy = (a_0 y + b_0 y_c) dt + \sigma_y^T dB,$$

$$\text{where } a_0 = \begin{pmatrix} -\alpha_I & 0 & 0 \\ 0 & -a & 0 \\ f_1 & f_2 & 0 \end{pmatrix}, b_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g_1 & g_2 & g_3 \end{pmatrix}, \text{ and } \sigma_y = \begin{pmatrix} \frac{-\eta\sigma_D}{\sqrt{2\eta - \eta^2}\sigma_D} & \pi & 0 \\ 0 & 0 & 0 \\ 0 & \sigma_U & 0 \end{pmatrix}.$$

Applying Ito's lemma to $\omega = X + U$ yields

$$d\omega = (f^T y + g^T y_c - aU) dt + \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \sigma_y^T dB. \quad (47)$$

By rearrangement, we then have

$$dy_c = (a_1 y + b_1 y_c) dt + \sigma_{y,c} dB,$$

where $a_1 = hf^T - ah(0, 1, 0) + \alpha_I m(1, 0, 0)$, $b_1 = hg^T + C$, $\sigma_{y,c} = (m\sigma_D + h\pi, \bar{0}, h\sigma_U)$ is a 3×3 matrix, and $\bar{0} = (0, 0, 0)^T$. Since y_c is observable, observing $d\omega$ is equivalent to observing

$$d\varsigma_1 = (f^T y - aU) dt + \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \sigma_y^T dB. \quad (48)$$

Since D is observable, observing dD is equivalent to observing

$$d\varsigma_2 = dD + \alpha_D D dt. \quad (49)$$

Let $\varsigma = (\varsigma_1, \varsigma_2)^T$. Stack y and y_c together and let $Z_Y = \begin{pmatrix} y \\ y_c \end{pmatrix}$. Conditional on the observation of ς , the posterior mean of Z_Y , denoted by $\widehat{Z}_Y = \begin{pmatrix} y_c \\ y_c \end{pmatrix}$ is given by³³

³²By verification, we will show that y_c indeed follows such a process.

³³See Lipster and Shirayayev (2001) and the Appendix A in Wang (1993).

$$\begin{aligned}
dy_c &= (a_0 + b_0)y_c dt + [\Sigma q_0^T + q_{1zs}] q_{ss}^{-1} d\bar{B}, \\
dy_c &= [(C + hg^T + hf^T)y_c - ah\hat{U} + \alpha_I m\hat{I}] dt + q_{2zs} q_{ss}^{-1} d\bar{B}.
\end{aligned} \tag{50}$$

where $q_0 = \begin{pmatrix} f_1 & f_2 - a & f_3 \\ \alpha_I & 0 & 0 \end{pmatrix}$, $q_{ss} = \begin{pmatrix} \pi^2 + \sigma_U^2 & \pi\sigma_D \\ \pi\sigma_D & \sigma_D^2 \end{pmatrix}$, $q_{1zs} = \begin{pmatrix} -\eta\sigma_D\pi & -\eta\sigma_D^2 \\ \pi^2 + \sigma_U^2 & \pi\sigma_D \\ 0 & 0 \end{pmatrix}$,
 $q_{2zs} = \begin{bmatrix} (h_1\pi + m_1\sigma_D)\pi + h_1\sigma_U^2 & (h_1\pi + m_1\sigma_D)\sigma_D \\ (h_2\pi + m_2\sigma_D)\pi + h_2\sigma_U^2 & (h_2\pi + m_2\sigma_D)\sigma_D \\ (h_3\pi + m_3\sigma_D)\pi + h_3\sigma_U^2 & (h_3\pi + m_3\sigma_D)\sigma_D \end{bmatrix}$, and $d\bar{B} = d\zeta - \begin{pmatrix} f^T y_c - a\hat{U} \\ \alpha_I \hat{I} \end{pmatrix} dt$.
Comparing the coefficients of the drift and the martingale of y_c yields

$$\begin{aligned}
0 &= \Sigma q_0^T + q_{1zs} - q_{2zs}, \\
0 &= C + hg^T + hf^T - ah(0, 1, 0) + \alpha_I m(1, 0, 0) - a_0 - b_0.
\end{aligned} \tag{51}$$

Rearrangement then yields

$$CC = \begin{bmatrix} -\alpha_I(1 + m_1) - h_1(f_1 + g_1) & -h_1(f_2 + g_2) + ah_1 & -g_3h_1 \\ -m_2\alpha_I - h_2(f_1 + g_1) & -a(1 - h_2) - h_2(f_2 + g_2) & -g_3h_2 \\ -m_3\alpha_I + (1 - h_3)(f_1 + g_1) & ah_3 + (1 - h_3)(f_2 + g_2) & g_3(1 - h_3) \end{bmatrix} \tag{52}$$

and h and m satisfy the equation

$$\begin{aligned}
&\begin{bmatrix} (h_1\pi + m_1\sigma_D)\pi + h_1\sigma_U^2 & (h_1\pi + m_1\sigma_D)\sigma_D \\ (h_2\pi + m_2\sigma_D)\pi + h_2\sigma_U^2 & (h_2\pi + m_2\sigma_D)\sigma_D \\ (h_3\pi + m_3\sigma_D)\pi + h_3\sigma_U^2 & (h_3\pi + m_3\sigma_D)\sigma_D \end{bmatrix} \\
&= \Sigma q_0^T + \begin{pmatrix} -\eta\sigma_D\pi & -\eta\sigma_D^2 \\ \pi^2 + \sigma_U^2 & \pi\sigma_D \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{53}$$

Let $O_t = E[(Z_{Y_t} - \hat{Z}_{Y_t})(Z_{Y_t} - \hat{Z}_{Y_t})^T]$ denote the variance-covariance matrix. By calculation, it is given by $O_t = \begin{pmatrix} \Sigma_t & 0 \\ 0 & 0 \end{pmatrix}$. We consider only a stationary Bayesian-Nash equilibrium and focus only on the steady state solution $dO_t = 0$. We will prove that the solution Σ is given by

$$0 = a_0\Sigma + \Sigma a_0^T + \Omega - [\Sigma q_0^T + q_{1zs}] (q_{ss})^{-1} [\Sigma q_0 + q_{1zs}]^T, \tag{54}$$

where $\Omega = \begin{pmatrix} 2\eta\sigma_D^2 & -\eta\pi\sigma_D & 0 \\ -\eta\pi\sigma_D & \pi^2 + \sigma_U^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Z_Y can be expressed as $Z_Y = \begin{pmatrix} y \\ y_c \end{pmatrix}$. Following Liptser and Shiryaev (2001), the steady state solution for $dO_t = 0$ is given by

$$\begin{aligned}
0 &= O \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \end{pmatrix}^T + \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \end{pmatrix} O + \begin{pmatrix} \Omega & \Omega_1 \\ \Omega_1^T & \Omega_2 \end{pmatrix} \\
&- \left[O \begin{pmatrix} q_0^T \\ 0 \end{pmatrix} + \begin{pmatrix} q_{1zs} \\ q_{2zs} \end{pmatrix} \right] (q_{ss})^{-1} \left[O \begin{pmatrix} q_0^T \\ 0 \end{pmatrix} + \begin{pmatrix} q_{1zs} \\ q_{2zs} \end{pmatrix} \right]^T,
\end{aligned} \tag{55}$$

where $\Omega_1 = \sigma_y^T \sigma_{y_c}$, $\Omega_2 = \sigma_{y_c} \sigma_{y_c}^T = (\pi h + \sigma_D m)(\pi h + \sigma_D m)^T + h h^T \sigma_U^2$, and $\sigma_{y_c} = \sigma_D m(1, 0, 0) + h(0, 1, 0) \sigma_y^T$.

Using partitioned matrices, we then have

$$\Sigma a_0^T + a_0 \Sigma + \Omega = [\Sigma q_0^T + q_{1zs}] (q_{ss})^{-1} [\Sigma q_0^T + q_{1zs}]^T, \quad (56)$$

$$0 = a_1 \Sigma + \Omega_1^T - [q_{2zs}] (q_{ss})^{-1} [\Sigma q_0^T + q_{1zs}]^T, \quad (57)$$

and

$$0 = \Omega_2 - (q_{2zs})(q_{ss})^{-1} q_{2zs}^T. \quad (58)$$

First we need to check that equations (57) and (58) hold. By matrix manipulation, it is easy to show that $q_{2zs}(q_{ss})^{-1} q_{2zs}^T = \sigma_{y_c} \sigma_{y_c}^T = (\pi h + \sigma_D m)(\pi h + \sigma_D m)^T + h h^T \sigma_U^2 = \Omega_2$. Hence, equation (58) holds. We also know that $a_1 \Sigma = (h, m) q_0 \Sigma$, $\Omega_1^T = (h, m) q_{1zs}^T$, and $[q_{2zs}] (q_{ss})^{-1} [\Sigma q_0^T + q_{1zs}]^T = [q_{2zs}] (q_{ss})^{-1} [q_{2zs}]^T = (h, m) q_{2zs}^T$. Equation (53) shows that $q_0 \Sigma + q_{1zs}^T - q_{2zs}^T = 0$. Therefore, Equation (57) holds. As a result, we know that Σ satisfies

$$\begin{aligned} 0 &= d\Sigma(t) = a_0 \Sigma + \Sigma a_0^T + \Omega + [\Sigma q_0^T + q_{1zs}] (q_{ss})^{-1} [\Sigma q_0^T + q_{1zs}]^T, \\ &= a_0 \Sigma + \Sigma a_0^T + \Omega + q_{2zs}(q_{ss})^{-1} q_{2zs}^T, \\ &= a_0 \Sigma + \Sigma a_0^T + \Omega - (\pi h + \sigma_D m)(\pi h + \sigma_D m)^T - h h^T \sigma_U^2. \end{aligned} \quad (59)$$

Since $\omega = \hat{U} + \hat{X}$, we have $d\omega = d\hat{U} + d\hat{X}$. From Lemma 1 in section 4, we know that $X - \hat{X} = -U + \hat{U}$. This leads to the solution

$$\Sigma = M^T \bar{\Sigma} M, \quad (60)$$

where $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ and $\bar{\Sigma}$ is given by

$$0 = a_0 \bar{\Sigma} + \bar{\Sigma} a_0^T + \bar{\Omega} - [\bar{\Sigma} q_0^T + q_1] (q_s)^{-1} [\bar{\Sigma} q_0^T + q_1]^T,$$

where $\bar{\Omega} = \begin{pmatrix} 2\eta\sigma_D^2 & -\eta\pi\sigma_D \\ -\eta\pi\sigma_D & \pi^2 + \sigma_U^2 \end{pmatrix}$, $q_0 = \begin{pmatrix} f_1 \\ f_2 - a - f_3 \end{pmatrix}$, $q_1 = \begin{pmatrix} -\eta\sigma_D\pi & -\eta\sigma_D^2 \\ \pi^2 + \sigma_U^2 & \pi\sigma_D \end{pmatrix}$, and $q_s = \begin{pmatrix} \pi^2 + \sigma_U^2 & \pi\sigma_D \\ \pi\sigma_D & \sigma_D^2 \end{pmatrix}$.

Applying Equation (53) yields

$$0 = a_0 \bar{\Sigma} + \bar{\Sigma} a_0^T + \bar{\Omega} - (\sigma_D \bar{m} + \pi \bar{h})(\sigma_D \bar{m} + \pi \bar{h})^T - \bar{h} \bar{h}^T \sigma_U^2, \quad (61)$$

where $\bar{h} = (h_1, h_2)^T$, $\bar{m} = (m_1, m_2)^T$, and $a_0 = \begin{pmatrix} -\alpha_I & 0 \\ 0 & -a \end{pmatrix}$. By rearrangement, $\bar{\Sigma} \equiv$

$\begin{pmatrix} \Sigma_{1,1} & \Sigma_{1,2} \\ \Sigma_{2,1} & \Sigma_{2,2} \end{pmatrix}$ is given by

$$\begin{aligned} \Sigma_{11} &= [2\eta\sigma_D^2 - h_1^2\sigma_U^2 - (h_1\pi + \sigma_D m_1)^2] / (2\alpha_I), \\ \Sigma_{12} &= -[\eta\pi\sigma_D + h_1 h_2 \sigma_U^2 + (\sigma_D m_1 + \pi h_1)(\sigma_D m_2 + \pi h_2)] / (a + \alpha_I), \\ \Sigma_{22} &= [\pi^2 + (1 - h_2^2)\sigma_U^2 - (h_2\pi + m_2\sigma_D)^2] / (2a). \end{aligned} \quad (62)$$

This completes the proof of Theorem 2.

We then prove Corollary 1 of this theorem. Since $d\bar{B}$ is a Brownian process with respect to \mathcal{F}_M , by orthogonalization, $d\bar{B} = q_{ss}^{\{1/2\}} d\bar{w}_M$. Hence,

$$\begin{aligned} dy_c &= A_M y_c dt + C_M d\bar{w}, \\ &= (a_0 + b_0) y_c dt + [\Sigma q_0^T + q_{1zs}] q_{ss}^{\{-1/2\}} d\bar{w}_M, \end{aligned} \quad (63)$$

where $d\bar{w}_M = q_{ss}^{\{-1/2\}} \left[d \begin{pmatrix} d\omega - g^T y_c dt \\ dD + \alpha_D D dt \end{pmatrix} - \begin{pmatrix} f^T y_c - a\hat{U} \\ \alpha_I \hat{I} \end{pmatrix} dt \right]$ and \bar{w}_M is a standard Wiener process with respect to $\mathcal{F}_M(t) = \mathcal{F}^{\{D, \omega\}}(t)$. In addition, the information structure generated by $\mathcal{F}^{\bar{w}}$ is equivalent to that generated by $\mathcal{F}_M(t) = \mathcal{F}^{\{D, \omega\}}(t)$.

By calculation,

$$\begin{aligned} dQ &= \frac{1}{\alpha_D + r} dD + \mu^T dy_c - r \left(\frac{\bar{D}}{r} + \frac{D - \bar{D}}{\alpha_D + r} + \mu^T y_c \right) dt + D dt \\ &= \left[\frac{\alpha_I}{\alpha_D + r} \hat{I} + \mu^T A_M y_c - r \mu^T \right] dt + \mu^T C_M d\bar{w}_M + \frac{1}{\alpha_D + r} d\bar{B} \\ &= b_M y_c dt + d_M d\bar{w}_M, \end{aligned}$$

where $b_M = \frac{\alpha_I}{\alpha_D + r} [1, 0, 0] + \mu^T A_M - r \mu^T$ and $d_M = \mu^T C_M + [0, \frac{1}{\alpha_D + r}] q_{ss}^{\{1/2\}}$.

Q.E.D.

B The Proof of Theorem 4.1

In this appendix, we solve the optimization problem of the informed trader. Proposition 3 gives the expression for dY , we conjecture that the value function of the informed trader is of the form:

$$J(W, Y, t) = -\exp \left[-\rho t - r\gamma W - S_0 - \frac{1}{2} Y^T L Y \right], \quad (64)$$

where L is a 5×5 symmetric matrix and S_0 is a constant. We then write the Bellman equation in the following form:

$$\begin{aligned} 0 &= \max_{C, \theta} -\exp(-\rho t - \gamma C) + J_t + \frac{1}{2} \text{tr}(J_{YY} C_Y C_Y^T) + J_W(rW - C - \frac{1}{2} k\theta^2 + \mu h_1 \theta + X q_0^T Y) \\ &\quad + J_Y^T (A_Y Y + B_Y \theta) + \frac{1}{2} J_{WW} X^2 q_1^T q_1 + J_{YW}^T C_Y X q_1, \end{aligned}$$

where $J_W = -r\gamma J$, $J_t = -\rho J$, $J_Y = -LYJ$, $J_{WW} = (r\gamma)^2 J$, $J_{YY} = (-L + LYY^T L)J$, and $J_{YW} = r\gamma LYJ$. q_0 , q_1 , A_Y , and C_Y are defined in Theorem 3.

The first-order condition (FOC) with respect to consumption yields

$$-J_W + \gamma \exp[-\rho t - \gamma C] = 0. \quad (65)$$

By rearrangement, we obtain

$$C = rW + \frac{1}{\gamma} \left[S_0 + \frac{1}{2} Y^T L Y - \log(r) \right]. \quad (66)$$

The first-order condition with respect to the order rate θ yields³⁴

$$-JY^T L B_Y - r\gamma J(\mu h_1 X - k\theta) = 0, \quad (67)$$

or

$$\theta = Y^T \frac{L B_Y + r\gamma \mu^T h \bar{1}_3}{r\gamma k}, \quad (68)$$

where $\bar{1}_3 = \begin{pmatrix} 0 & -1 & 0 & 1 & 1 \end{pmatrix}$.

Substituting the optimal consumption and optimal order rate into the Bellman equation yields

$$\begin{aligned} 0 &= (r - \rho) - \frac{1}{2} \text{tr}(C_Y^T L C_Y) + \left[r(S_0 + \frac{1}{2} Y^T L Y - \log r) - \frac{1}{2} r\gamma k \theta^2 - r\gamma X q_0^T Y \right] \\ &+ \frac{1}{2} (Y L C_Y C_Y^T L Y) - Y^T L A_Y Y + \frac{1}{2} (r\gamma)^2 X^2 q_1^T q_1 + r\gamma X Y^T L C_Y q_1, \end{aligned} \quad (69)$$

where tr denotes the trace of a matrix.

Note that this equation is the sum of two terms. The first is independent of Y and the second is a quadratic function of Y . A solution to this question is that these two terms are both zeros. We then have

$$(r - p) + r(S_0 - \log r) - \frac{1}{2} \text{tr}(C_Y^T L C_Y) = 0, \quad (70)$$

and

$$\begin{aligned} 0 &= Y^T \left[\frac{1}{2} r L - r\gamma \bar{1}_3 q_0^T + \frac{1}{2} (r\gamma)^2 (\bar{1}_3 \bar{1}_3^T) (q_1^T q_1) - L A_Y + \frac{1}{2} L C_Y C_Y^T L + r\gamma L C_Y q_1 \bar{1}_3^T \right] \\ &- \frac{1}{2r\gamma k} Y^T \left[(L B_Y + r\gamma \mu^T h \bar{1}_3) (L B_Y + r\gamma \mu^T h \bar{1}_3) \right] Y. \end{aligned} \quad (71)$$

Solving Equation (84) yields S_0 :

$$S_0 = \log r - \frac{(r - \rho)}{r} + \frac{1}{2r} \text{tr}(C_Y^T L C_Y). \quad (72)$$

Since the right side of Equation (71) is a scalar, it is equal to half of the sum of this expression and its transpose. L then satisfies the following Ricatti equation:

$$\begin{aligned} 0 &= rL - r\gamma(\bar{1}_3 q_0^T) - r\gamma(q_0 \bar{1}_3^T) - (A_Y^T L + L A_Y) \\ &+ L C_Y C_Y^T L - r\gamma k \left(\frac{L B_Y}{r\gamma k} + \frac{\mu h_1 \bar{1}_3}{k} \right) \left(\frac{L B_Y}{r\gamma k} + \frac{\mu h_1 \bar{1}_3}{k} \right)^T \\ &+ (r\gamma)^2 (q_1^T q_1) (\bar{1}_3 \bar{1}_3^T) + (r\gamma) L C_Y q_1 \bar{1}_3^T + (r\gamma) \bar{1}_3 q_1^T C_Y^T L. \end{aligned} \quad (73)$$

³⁴By calculation, the second-order condition with respect to the order rate for optimality is given by $-kJ_W < 0$.

Remark 1. The expression of the informed trader's position confirms our conjecture that it is a linear combination of the state variables I , U , \hat{I} , \hat{U} and \hat{X} .

Remark 2. When the informed trader just invests in bond, his value function is given by

$$J(W, t) = -\exp \left[-\rho t - rW - \log r - \frac{\rho - r}{r} \right].$$

Comparing the above value function with Equation (64) when the state variables are at their steady-state value, we then know that the certainty equivalent value to acquire the information process is given by $\frac{1}{r\gamma} \left[S_0 - \log r - \frac{\rho - r}{r} \right]$. By the relationship between perpetuity and annuity, the annualized certainty equivalent value of acquiring private information process is given by $\frac{1}{2r\gamma} \text{tr}(C_Y^T L C_Y)$.

The last step is to show that $\lim_{s \rightarrow \infty} E[J(W, Y, t + s) | \mathcal{F}_t] = 0$. By rearrangement, Equation (79) yields

$$-\exp \{ -(\rho t - \gamma C_t) \} = rJ(W, Y, t). \quad (74)$$

The Bellman equation is then given by

$$0 = rJ + E_t[dJ(W, Y, t)]. \quad (75)$$

Taking expectation with respect to t yields

$$\exp(-rs)E[J(W, Y, t + s) | \mathcal{F}_t] = J(W, Y, t). \quad (76)$$

Because $J(W, Y, t)$ is finite, we then have

$$\lim_{s \rightarrow \infty} \exp(-rs)E_t[J(W, Y, t + s)] = 0. \quad (77)$$

Q.E.D.

C The Proof of Theorem 4.2

In this appendix, we solve the optimization problem of the uninformed market makers. Corollary 1 in Theorem 2 gives the expression for dy_c . Let $Y_M \equiv y_c$. We conjecture that the value function of the market makers is of the form ³⁵

$$J(W_M, Y_M, t) = -\exp \left[-\rho t - r\gamma W_M - S_0 - \frac{1}{2} y_M^T L_M y_M \right], \quad (78)$$

where L_M is a 3×3 symmetric matrix and S_0 is a constant. We then write the Bellman equation in the following form:

$$\begin{aligned} 0 &= \max_{\bar{C}_M, y_M} -\exp(-\rho t - \gamma \bar{C}_M) + J_t + \frac{1}{2} \text{tr}(J_{\{Y_M Y_M\}} C_M C_M^T) + J_W(rW - \bar{C}_M + y_M b_M Y_M) \\ &+ J_{\{Y\}}^T (A_M Y_M) + \frac{1}{2} J_{WW} y_M^2 d_M d_M^T + J_{\{Y_M W\}}^T y_M C_M d_M^T, \end{aligned}$$

³⁵Since we consider only a symmetric equilibrium, we only need consider a representative market maker and omit the subscript i .

where $J_W = -r\gamma J$, $J_t = -\rho J$, $J_{\{Y_M\}} = -L_M Y_M J$, $J_{WW} = (r\gamma)^2 J$, $J_{\{Y_M Y_M\}} = (-L_M + L_M Y_M Y_M^T L_M) J$, and $J_{\{Y_M W\}} = r\gamma L_M Y_M J$. b_M , d_M , A_M , C_M are defined in Corollary 1 of Theorem 2, and tr denotes the trace of a matrix.

The first-order condition (FOC) with respect to consumption yields

$$-J_W + \gamma \exp[-\rho t - \gamma \bar{C}_M] = 0. \quad (79)$$

By rearrangement, we obtain

$$\bar{C}_M = rW_M + \frac{1}{\gamma} \left[S_0 + \frac{1}{2} Y_M^T L_M Y_M - \log(r) \right]. \quad (80)$$

The first-order condition with respect to y_M yields:

$$J_{\{W_M W_M\}} y_M (d_M d_M^T) + J_{\{Y_M W_M\}} C_M d_M^T + J_{\{W_M\}} b_M Y_M = 0, \quad (81)$$

³⁶ or

$$y_M \equiv f_M^T Y_M = -\frac{-b_M + d_M C_M^T L_M}{r\gamma d_M d_M^T} Y_M. \quad (82)$$

Substituting the optimal consumption and optimal position into the Bellman equation yields

$$\begin{aligned} 0 &= (r - \rho) - \frac{1}{2} tr(C_M^T L_M C_M) + \left[r(S_0 + \frac{1}{2} Y_M^T L_M Y_M - \log r) \right] \\ &+ \frac{1}{2} (Y_M^T L_M C_M C_M^T L_M Y_M) - Y_M^T L_M A_M Y_M - \frac{1}{2} (r\gamma)^2 y_M^2 d_M d_M^T, \end{aligned} \quad (83)$$

where tr denotes the trace of a matrix.

Note that this equation is the sum of two terms. The first is independent of Y and the second is a quadratic function of Y . The solution to this question is that these two terms are both zeros. We then have

$$(r - \rho) + r(S_0 - \log r) - \frac{1}{2} tr(C_M^T L_M C_M) = 0, \quad (84)$$

and

$$0 = rL_M + L_M C_M C_M^T L_M - L_M A_M - A_M^T L_M - (r\gamma)^2 d_M d_M^T f_M f_M^T, \quad (85)$$

Remark 1. The expression for a market maker's position confirms our conjecture that it is a linear combination of the state variables \hat{I} , \hat{U} and \hat{X} . Similar to the optimization problem of the informed trader, we can prove that $\lim_{s \rightarrow \infty} E[J(W_M, Y_M, t + s) | \mathcal{F}_{\{M, t\}}] = 0$.

Q.E.D.

³⁶By calculation, the second-order condition with respect to the order rate for optimality is given by $J_{\{W_M W_M\}}(d_M d_M^T) < 0$.

D The Proof of Proposition 1

From the expressions for dQ in equation (18), we have

$$\begin{aligned}\sigma_P^2 &= q_1^T \text{var}(dB, dB^T) q_1 \\ &= \left[\frac{\sigma_D}{r + \alpha_D} + \mu^T h \pi + \mu^T m \sigma_D \right]^2 + [\mu^T h \sigma_U]^2.\end{aligned}$$

Q. E. D.

E The Proof of Proposition 2

Let $Y_t \equiv \begin{pmatrix} I_t & U_t & \hat{I}_t & \hat{U}_t & \hat{X}_t \end{pmatrix}^T$. Multiplying the two sides of Equation (41) by $Y_{t+\tau}$ and taking the expectation yields

$$\Sigma_{\{Y,\tau\}} = e^{H\tau} \Sigma_{\{Y,\tau\}} e^{H\tau^T} + \Sigma_{\{\epsilon,\tau\}}. \quad (86)$$

Applying the Kronecker product yields

$$\text{Vec}(\Sigma_{\{Y,\tau\}}) = [I_{25} - (e^{H\tau} \otimes e^{H\tau})]^{-1} \text{Vec}(\Sigma_{\{\epsilon,\tau\}}), \quad (87)$$

where I_{25} is a 5×5 identity matrix.

Multiplying the two sides of Equation (41) by Y_t and taking the expectation yields

$$\begin{aligned}E[Y(t+\tau)Y(t)^T] &= A\Sigma_Y, \\ E[Y(t+\tau)Y(t-\tau)^T] &= A^2\Sigma_{\{Y,\tau\}},\end{aligned}$$

where $A = e^{H\tau}$.

Since $E[Y(t+\tau)Y(t)^T] = E[Y(t)Y(t-\tau)^T]$ and $X = \hat{X} + \hat{U} - U$, $\beta(\tau)$ is then given by

$$\beta(\tau) = \frac{\bar{1}_X^T \{A\Sigma_{\{Y,\tau\}} - A^2\Sigma_{\{Y,\tau\}} - \Sigma_{\{Y,\tau\}}\} \bar{1}_X}{\bar{1}_X^T \{2\Sigma_{\{Y,\tau\}} - A\Sigma_{\{Y,\tau\}} - \Sigma_{\{Y,\tau\}}A^T\} \bar{1}_X}, \quad (88)$$

where $\bar{1}_X = (0, -1, 0, 1, 1)^T$.

Q. E. D.

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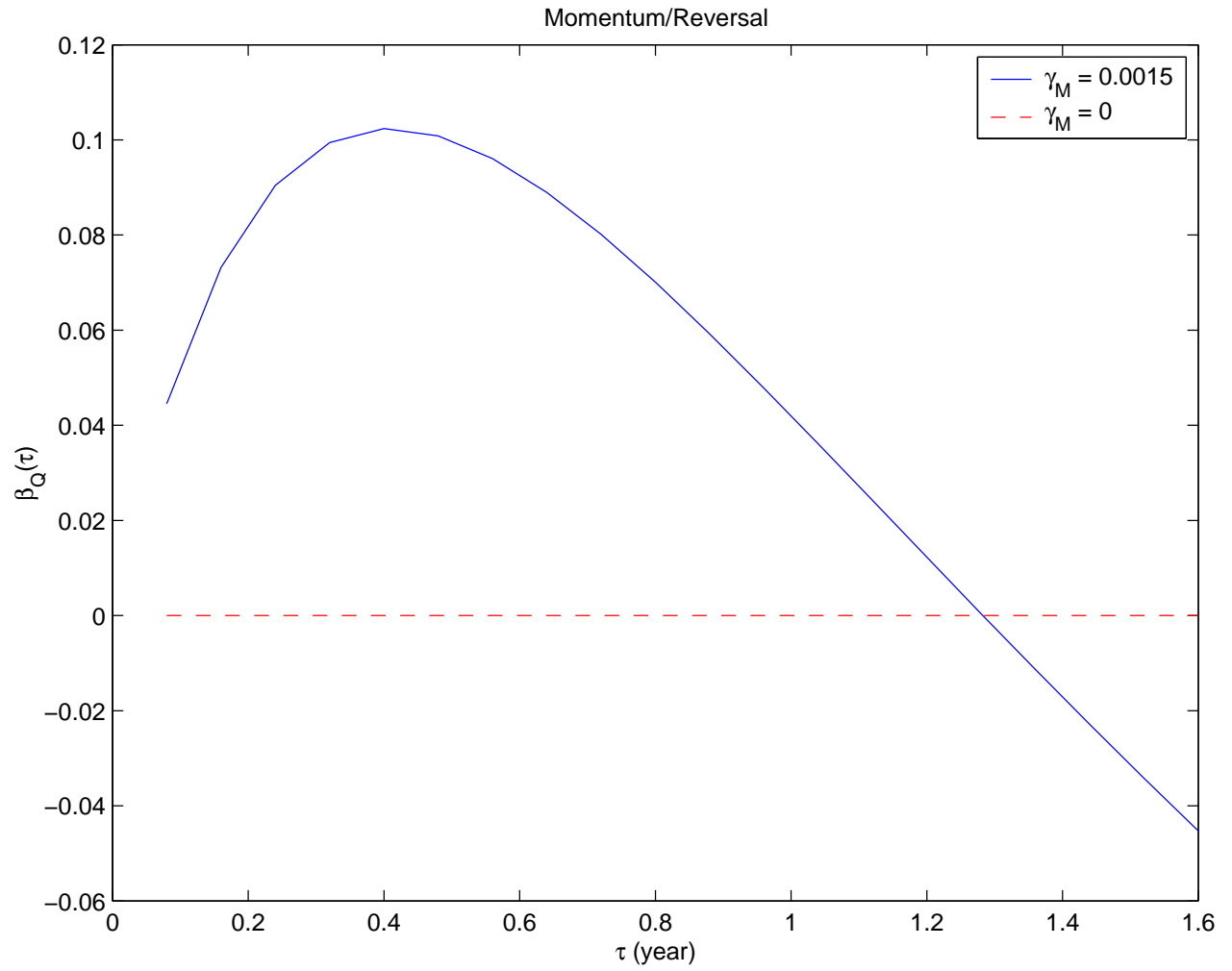


Figure 1: $\beta_Q(\tau)$ plotted against τ ($r = 0.04$, $\sigma_D = 0.5$, $\gamma = 0.000005$, $\sigma_U = 4000$, $a = 0.25$, $\alpha_I = 0.25$, $\eta = 0.8$, $\alpha_D = 0.000001$, $k = 0.01$, and $\pi = 0$)

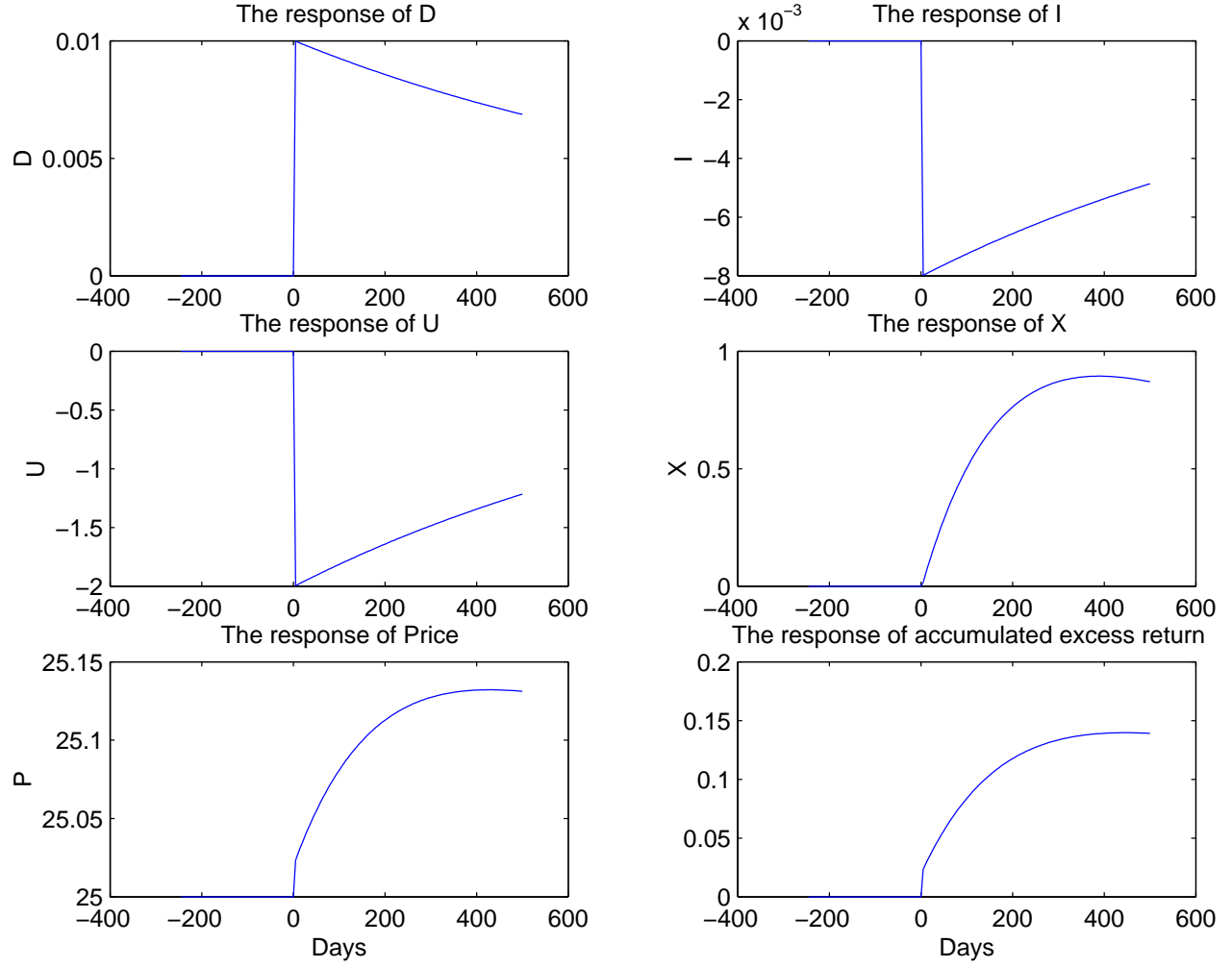


Figure 2: The responses to a dividend shock when market makers are risk-averse ($r = 0.04$, $\gamma = 0.000005$, $\gamma_M = 0.0015$, $\sigma_D = 0.5$, $\sigma_U = 4000$, $a = 0.25$, $\alpha_I = 0.25$, $\eta = 0.8$, $\alpha_D = 0.000001$, $k = 0.01$, $\bar{P} = 25$, and $\pi = -100$)

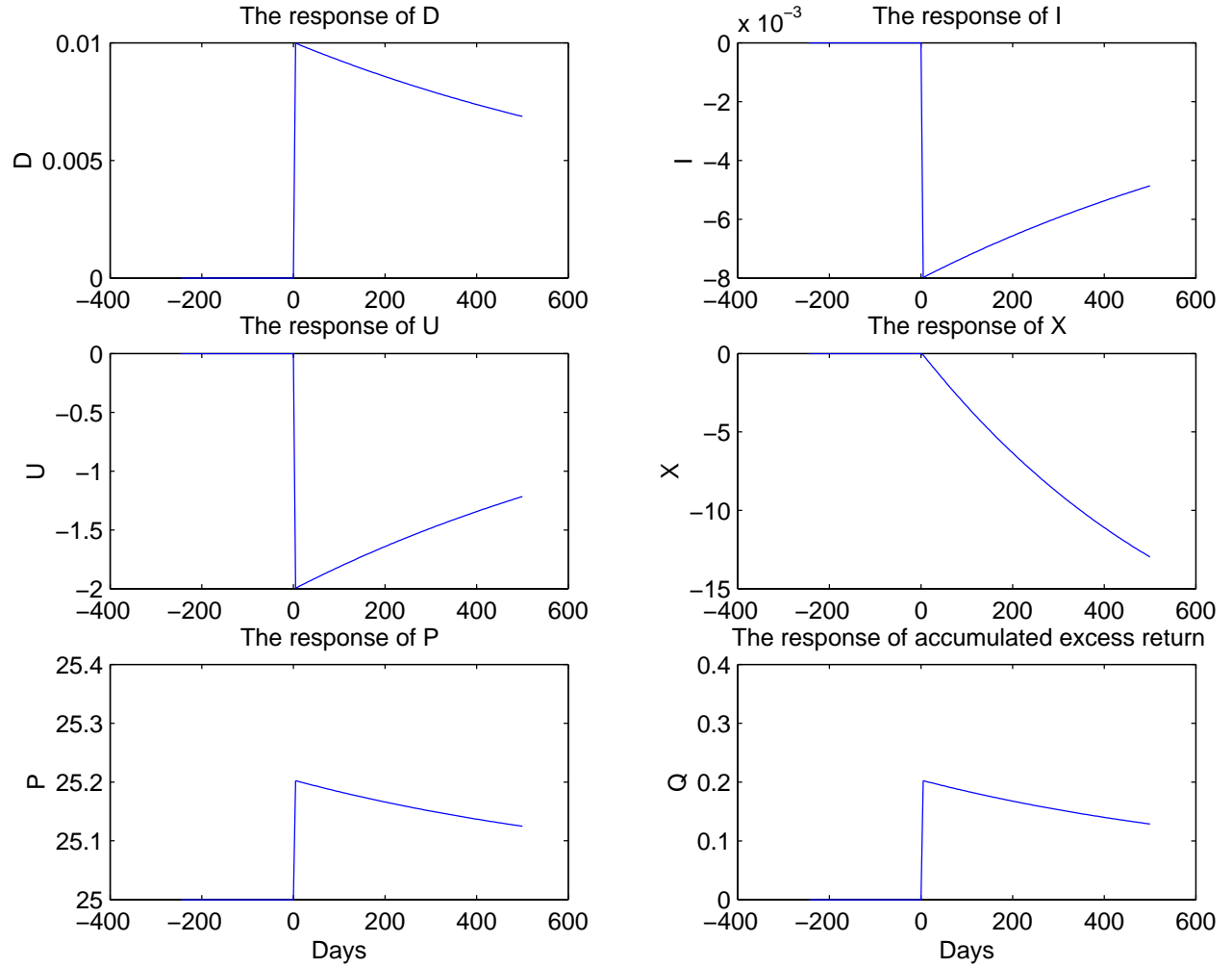


Figure 3: The responses to a dividend shock when market makers are risk-neutral ($r = 0.04$, $\gamma = 0.000005$, $\sigma_D = 0.5$, $\sigma_U = 4000$, $a = 0.25$, $\alpha_I = 0.25$, $\eta = 0.8$, $\alpha_D = 0.000001$, $k = 0.01$, $\bar{P} = 25$, and $\pi = -100$)

Figure 4: The market impact cost, instantaneous return variance and the trading volume of the informed trader ($r = 0.04$, $\gamma = 0.001$, $\gamma_M = 0.0001$, $\sigma_U = 40000$, $a = 4$, $\alpha_I = 4$, $\eta = 0.1$, $\alpha_D = 0.000001$, $k = 0.001$, and $\pi = 0$)

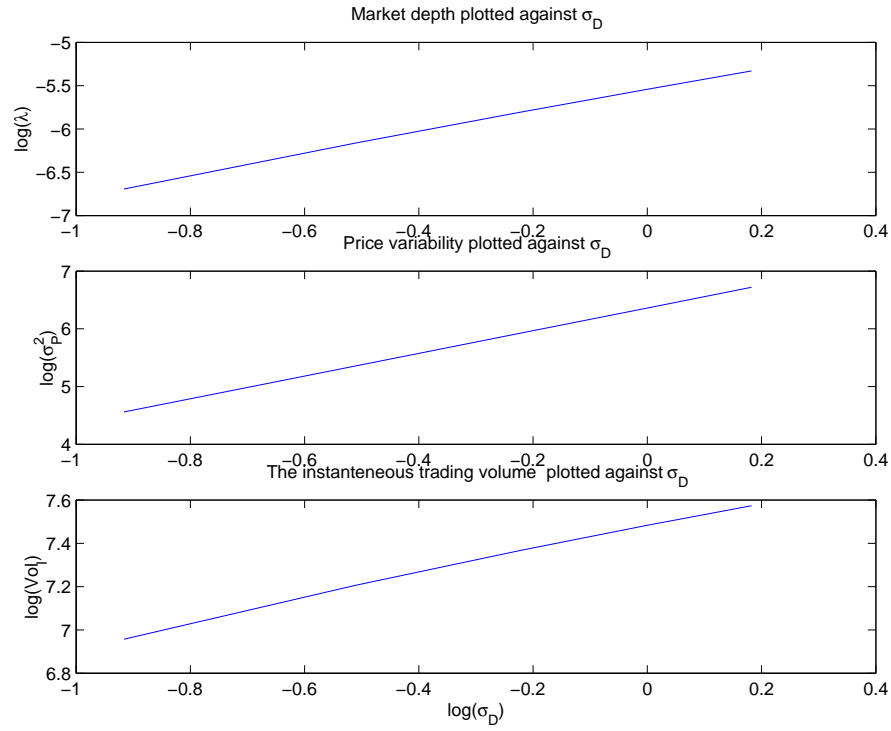


Figure 5: $\beta(\tau)$ plotted against τ ($r = 0.04$, $\sigma_D = 0.5$, $\sigma_U = 40000$, $a = 4$, $\alpha_I = 4$, $\eta = 0.1$, $\alpha_D = 0.000001$, $k = 0.001$, and $\pi = 0$)

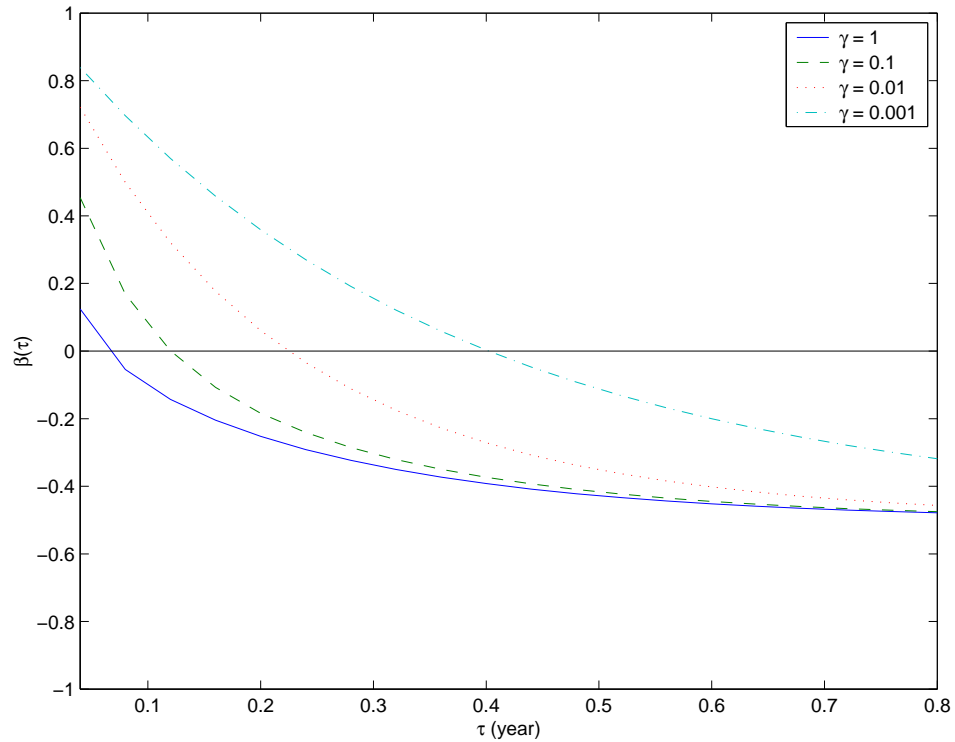


Figure 6: $\beta(\tau)$ plotted against τ ($r = 0.04$, $\sigma_D = 0.5$, $\sigma_U = 40000$, $a = 4$, $\alpha_I = 4$, $\eta = 0.1$, $\alpha_D = 0.000001$, $\gamma = 0.001$, and $\pi = 0$)

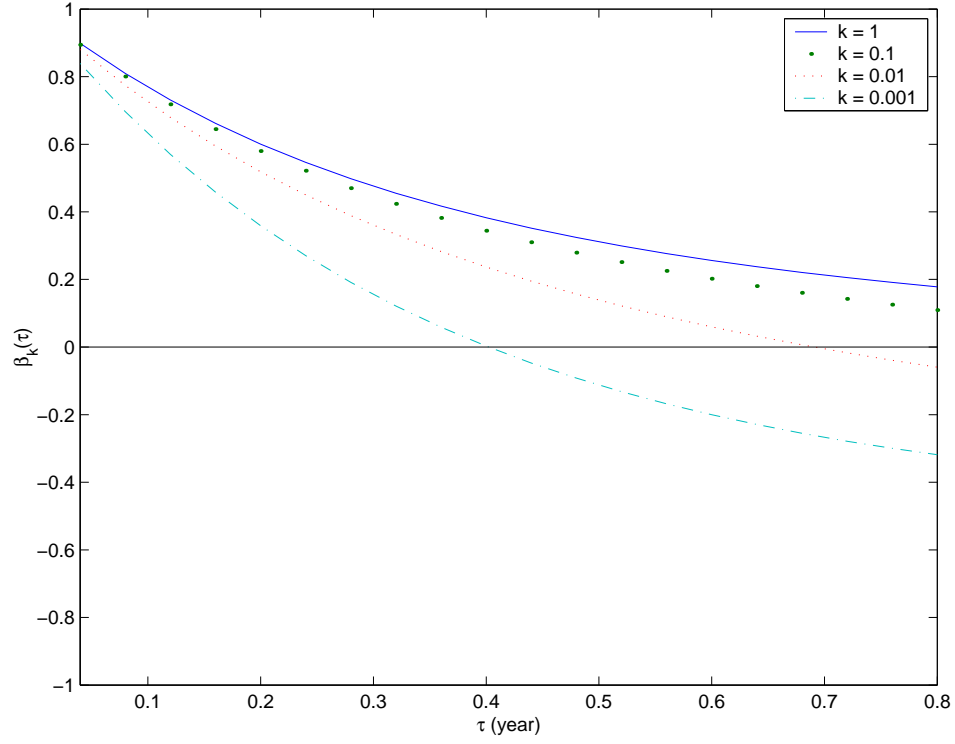


Figure 7: $\beta_p(\tau)$ plotted against τ ($r = 0.04$, $\sigma_D = 0.5$, $\sigma_U = 40000$, $a = 4$, $\alpha_I = 4$, $\eta = 0.1$, $\alpha_D = 0.000001$, $k = 0.001$, and $\pi = 0$)

