Table 1: Approximate Market Shares, California CARB Gasoline Post-Merger Numbers in Parentheses					
Company	i	Refining Market Share (σ_i)	Refining Capital Share	Retail Market Share (s_i)	Retail Capital Share
Chevron	1	26.4 (26.6)	29.5 (29.5)	19.2 (19.5)	19.0 (19.0)
Tosco	2	21.5 (21.7)	21.7 (21.7)	17.8 (18.0)	17.8 (17.8)
Equilon	3	16.6 (16.7)	16.1 (16.1)	16.0 (16.2)	16.0 (16.0)
Arco	4	13.8 (13.9)	13.0 (13.0)	20.4 (20.7)	22.0 (22.0)
Mobil	5	7.0 (13.3)	6.2 (12.4)	9.7 (17.5)	9.3 (17.8)
Exxon	6	7.0 (0.0)	6.2 (0.0)	8.9 (0.0)	8.5 (0.0)
Ultramar	7	5.4 (5.4)	4.7 (4.7)	6.8 (6.9)	6.4 (6.4)
Paramount	8	2.3 (2.3)	2.0 (2.0)	0.0 (0.0)	0.0 (0.0)
Kern	9	0.0 (0.0)	0.0 (0.0)	0.3 (0.3)	0.27 (0.27)
Koch	10	0.0 (0.0)	0.0 (0.0)	0.2 (0.2)	0.18 (0.18)
Vitol	11	0.0 (0.0)	0.0 (0.0)	0.2 (0.2)	0.18 (0.18)
Tesoro	12	0.0 (0.0)	0.0 (0.0)	0.2 (0.2)	0.18 (0.18)
PetroDiamond	13	0.0 (0.0)	0.0 (0.0)	0.1 (0.1)	0.09 (0.09)
Time	14	0.0 (0.0)	0.0 (0.0)	0.1 (0.1)	0.09 (0.09)
Glencoe	15	0.0 (0.0)	0.0 (0.0)	0.1 (0.1)	0.09 (0.09)

Table 2: Markups and Quantity Reduction, in Percent, for a Symmetric Industry											
Number of Firms	2	3	4	5	6	7	8	9	10	15	20
Average Markup	74.0	42.3	30.0	22.7	18.4	15.5	13.4	11.8	10.5	6.8	5.1
Q/Q_f, in percent	82.9	87.7	92.2	94.2	95.4	96.2	96.8	97.2	97.5	98.4	98.8

Table 3: Analysis of Exxon – Mobil Merger								
Cases			Markup as a Percent of Retail Price (Quantity as a Percent of Fully Efficient Quantity in Parentheses)					
α	β	η	Symmetric	Balanced Asymmetric	Pre-merger Markup	Post-merger Markup	Refinery Sale	Retail Sale
1/3	5	½	6.9 (98.4)	18.4 (95.3)	20.0 (94.6)	21.3 (94.3)	20.1 (94.6)	21.2 (94.3)
1/5	5	½	7.9 (98.7)	21.6 (96.0)	23.6 (95.4)	25.2 (95.2)	23.7 (95.4)	25.2 (95.2)
1/3	3	½	7.0 (98.4)	18.7 (95.3)	20.3 (94.6)	21.7 (94.3)	20.5 (94.6)	21.6 (94.4)
1/3	5	1/3	8.7 (98.2)	23.0 (94.6)	25.1 (93.8)	26.7 (93.5)	25.2 (93.8)	26.7 (93.5)

Table 4: Analysis of Exxon – Mobil Merger					
Cases			Expected Percentage Quantity Decrease (Percentage Price Increase in Parentheses)		
α	β	η	Full Merger	Refinery Sale	Retail Sale
1/3	5	½	0.31 (0.94)	0.03 (0.09)	0.30 (0.90)
1/5	5	½	0.27 (1.36)	0.02 (0.11)	0.25 (1.29)
1/3	3	½	0.32 (0.97)	0.05 (0.15)	0.30 (0.89)
1/3	5	1/3	0.35 (1.06)	0.03 (0.08)	0.34 (1.03)

Appendix

Proof of Theorems 1 and 3:

Before proceeding, note that differentiating the equilibrium condition in equation (7) implies that

$$\left(\frac{K}{Q}\right)\frac{\partial Q}{\partial K}=\frac{\varepsilon^{-1}}{\varepsilon^{-1}+\eta^{-1}},\ \left(\frac{\Gamma}{Q}\right)\frac{\partial Q}{\partial \Gamma}=\frac{\eta^{-1}}{\varepsilon^{-1}+\eta^{-1}},\ and\ \left(\frac{K}{P}\right)\frac{\partial P}{\partial K}=-\left(\frac{\Gamma}{P}\right)\frac{\partial P}{\partial \Gamma}=\frac{(\eta\varepsilon)^{-1}}{\varepsilon^{-1}+\eta^{-1}}.$$

Differentiating the firm's profit function in equation (8) and substituting the relations given above yields the following first order conditions:

$$\frac{\partial \pi_i}{\partial \hat{k}_i}=\left(v'\left(\frac{s_iQ}{k_i}\right)-p\right)\left(Q\left(\frac{1}{K}-\frac{\hat{k}_i}{K^2}\right)+s_i\frac{\partial Q}{\partial K}\right)+\left(p-c'\left(\frac{\sigma_iQ}{\gamma_i}\right)\right)\left(\sigma_i\frac{\partial Q}{\partial K}\right)-Q(s_i-\sigma_i)\frac{\partial p}{\partial K}$$

$$=\frac{Q}{K}\left[(v_i'-p)\left((1-s_i)+s_i\frac{\varepsilon^{-1}}{\varepsilon^{-1}+\eta^{-1}}\right)+(p-c_i')\left(\sigma_i\frac{\varepsilon^{-1}}{\varepsilon^{-1}+\eta^{-1}}\right)-p(s_i-\sigma_i)\frac{(\eta\varepsilon)^{-1}}{\varepsilon^{-1}+\eta^{-1}}\right].$$

Similarly,

$$\frac{\partial \pi_i}{\partial \hat{\gamma}_i}=\left(v'\left(\frac{s_iQ}{k_i}\right)-p\right)\left(s_i\frac{\partial Q}{\partial \Gamma}\right)+\left(p-c'\left(\frac{\sigma_iQ}{\gamma_i}\right)\right)\left(Q\left(\frac{1}{\Gamma}-\frac{\hat{\gamma}_i}{\Gamma^2}\right)+\sigma_i\frac{\partial Q}{\partial \Gamma}\right)-Q(s_i-\sigma_i)\frac{\partial p}{\partial \Gamma}$$

$$=\frac{Q}{\Gamma}\left[(v_i'-p)\left(s_i\frac{\eta^{-1}}{\varepsilon^{-1}+\eta^{-1}}\right)+(p-c_i')\left(1-\sigma_i+\sigma_i\frac{\eta^{-1}}{\varepsilon^{-1}+\eta^{-1}}\right)+p(s_i-\sigma_i)\frac{(\eta\varepsilon)^{-1}}{\varepsilon^{-1}+\eta^{-1}}\right].$$

Thus,

$$\frac{K}{Q}\frac{\partial \pi_i}{\partial \hat{k}_i}+\frac{\Gamma}{Q}\frac{\partial \pi_i}{\partial \hat{\gamma}_i}=v_i'-c_i'.$$

In an interior equilibrium, then, $v_i'-c_i'=0$. With either of the first order conditions, we obtain

(15).

The Case of s_i=0, σ_i>0.

If σ_i>0, the first order condition on $\hat{\gamma}_i$ holds with equality. Consequently, using s_i=0,

$$0 = \frac{Q}{\Gamma}\left[(p - c_i')\left(1 - \sigma_i + \sigma_i \frac{\eta^{-1}}{\varepsilon^{-1} + \eta^{-1}}\right) + p(0 - \sigma_i)\frac{(\eta\varepsilon)^{-1}}{\varepsilon^{-1} + \eta^{-1}} \right]$$

This yields

$$\frac{p - c_i'}{p} = \frac{\sigma_i}{\varepsilon + \eta(1 - \sigma_i)},$$

a formula that respects (15) (for s_i=0). In addition, we have

$$0 \geq \left. \frac{\partial \pi_i}{\partial \hat{k}_i} \right|_{s_i = 0},$$

which gives

$$\frac{v'(0) - p}{p} \leq \frac{-\sigma_i}{\varepsilon + \eta(1 - \sigma_i)}.$$

Summarizing,

$$\frac{v_i' - p}{p} \leq \frac{s_i - \sigma_i}{\varepsilon(1 - s_i) + \eta(1 - \sigma_i)}, \textit{ with equality if } s_i > 0.$$

and

$$\frac{p - c_i'}{p} \leq \frac{s_i - \sigma_i}{\varepsilon(1 - s_i) + \eta(1 - \sigma_i)}, \textit{ with equality if } \sigma_i > 0.$$

Suppose k_i=0. Then $kv(q/k) = \frac{v(q/k)}{1/k} \approx qv'(q/k) \xrightarrow[k \to 0]{} 0$. Thus, an agent with k_i=0 will report $\hat{k}_i = 0$. Similarly, an agent with γ_i=0 produces zero. This yields (13) and (14). Q.E.D.

Proof of Theorem 4: It is readily checked that the following substitutions hold, even when a share is zero.

$$\sum_{i=1}^{n} v_i' s_i - \sum_{i=1}^{n} c_i' \sigma_i = \sum_{i=1}^{n} (v_i' - p) s_i + \sum_{i=1}^{n} (p - c_i') \sigma_i$$

$$= \sum_{i=1}^{n} (v_i' - p) s_i + \sum_{i=1}^{n} (p - v_i') \sigma_i = \sum_{i=1}^{n} (v_i' - p) s_i - \sum_{i=1}^{n} (v_i' - p) \sigma_i$$

$$= \sum_{i=1}^{n} (v_i' - p)(s_i - \sigma_i) = \sum_{i=1}^{n} \frac{p(s_i - \sigma_i)^2}{\varepsilon(1 - s_i) + \eta(1 - \sigma_i)}.$$ Q.E.D.

Proof of Theorem 5: Applying (7), (15), (17):

$$\left(\frac{s_i Q}{k_i}\right)^{-\frac{1}{\varepsilon}} = v'(\frac{s_i Q}{k_i}) = p\left(1+\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)}\right) = v'\left(\frac{Q}{\sum_{i=1}^{n}\hat{k}_i}\right)\left(1+\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)}\right).$$

This readily gives the first part of (18); the second half is symmetric.

Rewrite (18) to obtain

$$k_i = \hat{k}_i\left(1+\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)}\right)^{\varepsilon}.$$

Thus

$$\frac{k_i}{\sum_{j=1}^{n}\hat{k}_j} = s_i\left(1+\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)}\right)^{\varepsilon}.$$

Thus

$$\frac{\sum_{i=1}^{n}k_i}{\sum_{i=1}^{n}\hat{k}_i} = \sum_{i=1}^{n}s_i\left(1+\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)}\right)^{\varepsilon}.$$

A similar calculation gives

$$\frac{\sum_{i=1}^{n}\gamma_i}{\sum_{i=1}^{n}\hat{\gamma}_i} = \sum_{i=1}^{n}\sigma_i\left(1+\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)}\right)^{-\eta}.$$

Note that, with constant elasticity, actual quantity is

$$Q = \left(\sum_{i=1}^{n}\hat{k}_i\right)^{\frac{\eta}{\varepsilon+\eta}}\left(\sum_{i=1}^{n}\hat{\gamma}_i\right)^{\frac{\varepsilon}{\varepsilon+\eta}},$$

and

$$Q_f = \left(\sum_{i=1}^{n}k_i\right)^{\frac{\eta}{\varepsilon+\eta}}\left(\sum_{i=1}^{n}\gamma_i\right)^{\frac{\varepsilon}{\varepsilon+\eta}}.$$

Substitution gives (18). One obtains (19) from

$$\frac{p}{p_f}=\frac{v'(Q/\sum_{i=1}^n \hat{k}_i)}{v'(Q_f/\sum_{i=1}^n k_i)}=\left(\frac{Q}{Q_f}\frac{\sum_{i=1}^n k_i}{\sum_{i=1}^n \hat{k}_i}\right)^{\frac{-1}{\varepsilon}}=\left(\frac{Q_f}{Q}\frac{\sum_{i=1}^n \hat{k}_i}{\sum_{i=1}^n k_i}\right)^{\frac{1}{\varepsilon}}$$

$$=\left[\left(\frac{\sum_{i=1}^n k_i}{\sum_{i=1}^n \hat{k}_i}\right)^{\frac{\eta}{\varepsilon+\eta}}\left(\frac{\sum_{i=1}^n \gamma_i}{\sum_{i=1}^n \hat{\gamma}_i}\right)^{\frac{\varepsilon}{\varepsilon+\eta}}\frac{\sum_{i=1}^n \hat{k}_i}{\sum_{i=1}^n k_i}\right]^{\frac{1}{\varepsilon}}=\left[\left(\frac{\sum_{i=1}^n \gamma_i}{\sum_{i=1}^n \hat{\gamma}_i}\right)\left(\frac{\sum_{i=1}^n \hat{k}_i}{\sum_{i=1}^n k_i}\right)\right]^{\frac{1}{\varepsilon+\eta}}$$

$$=\left[\frac{\sum_{i=1}^n \sigma_i\left(1+\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)}\right)^{-\eta}}{\sum_{i=1}^n s_i\left(1+\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)}\right)^{\varepsilon}}\right]^{\frac{1}{\varepsilon+\eta}}.$$

Q.E.D.

Analysis of Section 4:

Using the market calculations (23) and (24), rewrite profits to obtain

$$\pi_i=r(Q)q_i-k_i w\left(q_i/k_i\right)-\gamma_i c\left(x_i/\gamma_i\right)-p(q_i-x_i)$$

$$=(r(Q)-p)q_i-k_i w\left(q_i/k_i\right)+px_i-\gamma_i c\left(x_i/\gamma_i\right)$$

$$=w'\left(Q/K\right)q_i-k_i w\left(q_i/k_i\right)+c'\left(Q/\Gamma\right)x_i-c\left(x_i/\gamma_i\right)$$

$$=w'\left(Q/K\right)\frac{\hat{k}_i}{K}Q-k_i w\left(\frac{\hat{k}_i}{K}\frac{Q}{k_i}\right)+c'\left(Q/\Gamma\right)\frac{\hat{\gamma}_i}{\Gamma}Q-\gamma_i c\left(\frac{\hat{\gamma}_i}{\Gamma}\frac{Q}{\gamma_i}\right).$$

The equilibrium quantity is given by

$$r(Q)-w'(Q/K)-c'(Q/\Gamma)=0.$$

From this equation, and applying (25), it is a routine computation to show:

$$\frac{K}{Q}\frac{dQ}{dK}=\frac{K}{Q}\frac{-(w'')Q/K^2}{r'-w''/K-c''/\Gamma}=\frac{(r-p)\beta^{-1}}{r\alpha^{-1}+(r-p)\beta^{-1}+p\eta^{-1}}=\frac{B}{A+B+C}.$$

Similarly,

$$\frac{\Gamma}{Q}\frac{dQ}{d\Gamma} = \frac{C}{A+B+C}.$$

Differentiating π_i, and using the analogous notation for

$$0 = \frac{K}{Q}\frac{\partial \pi_i}{\partial \hat{k}_i} = (w' - w_i')(1 - s_i + s_i \frac{K}{Q}\frac{dQ}{dK}) - w'' \frac{s_i Q}{K} + (c' - c_i')\sigma_i \frac{K}{Q}\frac{dQ}{dK} + (w'' s_i \frac{Q}{K} + c'' \sigma_i \frac{Q}{\Gamma})\frac{K}{Q}\frac{dQ}{dK}$$

$$= (w' - w_i')(1 - s_i + s_i \frac{K}{Q}\frac{dQ}{dK}) + (c' - c_i')\sigma_i \frac{K}{Q}\frac{dQ}{dK} - \beta^{-1} w' s_i + (\beta^{-1} w' s_i + \eta^{-1} c' \sigma_i)\frac{K}{Q}\frac{dQ}{dK}$$

$$= (w' - w_i')(1 - s_i + s_i \frac{K}{Q}\frac{dQ}{dK}) + (c' - c_i')\sigma_i \frac{K}{Q}\frac{dQ}{dK} - r\left(Bs_i - (Bs_i + C\sigma_i)\frac{K}{Q}\frac{dQ}{dK}\right)$$

$$= (w' - w_i')(1 - s_i + s_i \frac{K}{Q}\frac{dQ}{dK}) + (c' - c_i')\sigma_i \frac{K}{Q}\frac{dQ}{dK} - r\left(Bs_i\left(1 - \frac{K}{Q}\frac{dQ}{dK}\right) - C\sigma_i \frac{K}{Q}\frac{dQ}{dK}\right)$$

Similarly, and symmetrically,

$$0 = \frac{\Gamma}{Q}\frac{\partial \pi_i}{\partial \hat{\gamma}_i} = (w' - w_i')s_i \frac{\Gamma}{Q}\frac{dQ}{d\Gamma} + (c' - c_i')(1 - \sigma_i + \sigma_i \frac{\Gamma}{Q}\frac{dQ}{d\Gamma}) - r\left(C\sigma_i\left(1 - \frac{\Gamma}{Q}\frac{dQ}{d\Gamma}\right) - Bs_i \frac{\Gamma}{Q}\frac{dQ}{d\Gamma}\right).$$

These equations can be expressed, substituting the elasticities with respect to capacity, as

$$\begin{bmatrix} A+B+C-s_i(A+C) & B\sigma_i \\ Cs_i & A+B+C-\sigma_i(A+B) \end{bmatrix}\begin{pmatrix} (w'-w_i')/r \\ (c'-c_i')/r \end{pmatrix} = \begin{pmatrix} Bs_i(A+C) - BC\sigma_i \\ c\sigma_i(A+B) - BCs_i \end{pmatrix}$$

The determinant of the left hand side matrix is given by

$$DET = [A+B+C-s_i(A+C)][A+B+C-\sigma_i(A+B)] - BCs_i\sigma_i$$

$$= (A+B+C)[(A+B+C)(1-s_i)(1-\sigma_i) + Bs_i(1-\sigma_i) + C(1-s_i)\sigma_i]$$

$$= (A+B+C)[A(1-s_i)(1-\sigma_i) + B(1-\sigma_i) + C(1-s_i)].$$

Thus,

$$\begin{pmatrix} (w' - w_i')/r \\ (c' - c_i')/r \end{pmatrix} = \frac{1}{DET}\begin{bmatrix} A+B+C-\sigma_i(A+B) & -B\sigma_i \\ -Cs_i & A+B+C-s_i(A+C) \end{bmatrix}\begin{pmatrix} Bs_i(A+C) - BC\sigma_i \\ c\sigma_i(A+B) - BCs_i \end{pmatrix}$$

$$= \frac{1}{DET}\begin{pmatrix} (A+B+C-\sigma_i(A+B))(Bs_i(A+C)-BC\sigma_i) - B\sigma_i(c\sigma_i(A+B)-BCs_i) \\ -Cs_i(Bs_i(A+C)-BC\sigma_i) + (A+B+C-s_i(A+C))(c\sigma_i(A+B)-BCs_i) \end{pmatrix}$$

$$= \frac{A+B+C}{DET}\begin{pmatrix} (Bs_i(A+C)-BC\sigma_i) - ABs_i\sigma_i \\ (C\sigma_i(A+B)-BCs_i) - ACs_i\sigma_i \end{pmatrix} = \frac{A+B+C}{DET}\begin{pmatrix} B[C(s_i-\sigma_i)+As_i(1-\sigma_i)] \\ C[B(\sigma_i-s_i)+A\sigma_i(1-s_i)] \end{pmatrix}.$$

Thus,

$$MHI = \sum_{i=1}^{n}\left(\frac{(r(Q)-p-w_i')s_i + (p-c_i')\sigma_i}{r}\right)$$

$$= \sum_{i=1}^{n}\left(\frac{(w'-w_i')s_i + (c'-c_i')\sigma_i}{r}\right)$$

$$= \sum_{i=1}^{n}\left[\left(\frac{A+B+C}{DET}\right)\left(s_iB[C(s_i-\sigma_i)+As_i(1-\sigma_i)] + \sigma_iC[B(\sigma_i-s_i)+A\sigma_i(1-s_i)]\right)\right]$$

$$= \sum_{i=1}^{n}\left[\left(\frac{A+B+C}{DET}\right)\left[BC(s_i-\sigma_i)^2 + ABs_i^2(1-\sigma_i) + AC\sigma_i^2(1-s_i)\right]\right]$$

$$= \sum_{i=1}^{n}\frac{BC(s_i-\sigma_i)^2 + ABs_i^2(1-\sigma_i) + AC\sigma_i^2(1-s_i)}{A(1-s_i)(1-\sigma_i) + B(1-\sigma_i) + C(1-s_i)}.$$

Q.E.D.

Derivation of the optimal quantity formula with constant elasticities

$$\begin{pmatrix} w' - w_i' \\ c' - c_i' \end{pmatrix} = r\begin{pmatrix} \dfrac{B[C(s_i-\sigma_i)+As_i(1-\sigma_i)]}{A(1-s_i)(1-\sigma_i)+B(1-\sigma_i)+C(1-s_i)} \\ \dfrac{C[B(\sigma_i-s_i)+A\sigma_i(1-s_i)]}{A(1-s_i)(1-\sigma_i)+B(1-\sigma_i)+C(1-s_i)} \end{pmatrix} \equiv r\begin{pmatrix} \psi_i \\ \chi_i \end{pmatrix}.$$

Then

$$\left(\frac{s_i Q}{k_i}\right)^{\frac{1}{\beta}} = w_i' = w' - r\psi_i = r(1-\theta-\psi_i)$$

or,

$$k_i = s_i Q[r(1-\theta-\psi_i)]^{-\beta}$$

Similarly,

$$\gamma_i = \sigma_i Q[r(\theta-\chi_i)]^{-\eta}$$

The efficient solution satisfies $r - w'\left(Q_f \big/ \sum k_i\right) - c'\left(Q_f \big/ \sum \gamma_i\right) = 0$, or

$$0 = Q_f^{-\frac{1}{\alpha}} - \left(\frac{Q_f}{Q}\right)^{\frac{1}{\beta}} r\left(\sum_{i=1}^{n} s_i[1-\theta-\psi_i]^{-\beta}\right)^{\frac{-1}{\beta}} - \left(\frac{Q_f}{Q}\right)^{\frac{1}{\eta}} r\left(\sum_{i=1}^{n} \sigma_i[\theta-\chi_i]^{-\eta}\right)^{\frac{-1}{\eta}}$$

Thus, substituting and dividing by $r=Q^{-1/\alpha}$

$$0 = \left(\frac{Q_f}{Q}\right)^{\frac{-1}{\alpha}} - \left(\frac{Q_f}{Q}\right)^{\frac{1}{\beta}} \left(\sum_{i=1}^{n} s_i[1-\theta-\psi_i]^{-\beta}\right)^{\frac{-1}{\beta}} - \left(\frac{Q_f}{Q}\right)^{\frac{1}{\eta}} \left(\sum_{i=1}^{n} \sigma_i[\theta-\chi_i]^{-\eta}\right)^{\frac{-1}{\eta}}$$

or

$$1 = \left(\frac{Q_f}{Q}\right)^{A+\frac{1}{\beta}} \left(\sum_{i=1}^{n} s_i[1-\theta-\psi_i]^{-\beta}\right)^{\frac{-1}{\beta}} + \left(\frac{Q_f}{Q}\right)^{A+\frac{1}{\eta}} \left(\sum_{i=1}^{n} \sigma_i[\theta-\chi_i]^{-\eta}\right)^{\frac{-1}{\eta}}.$$

or

$$1 = \left(\frac{Q}{Q_f}\right)^{-\left(A+\frac{1}{\beta}\right)} \left(\sum_{i=1}^{n} s_i[1-\theta-\psi_i]^{-\beta}\right)^{\frac{-1}{\beta}} + \left(\frac{Q}{Q_f}\right)^{-\left(A+\frac{1}{\eta}\right)} \left(\sum_{i=1}^{n} \sigma_i[\theta-\chi_i]^{-\eta}\right)^{\frac{-1}{\eta}}.$$

This equation can be solved for the ratio $Q_f \big/ Q$ which yields the underproduction.

Analysis with a Competitive Fringe

In many circumstances, there are firms that are best modeled as price-takers. Moreover, in some instances, the competitive fringe may use a distinct production technology, and therefore have different cost elasticities. This section replicates the general model with a competitive fringe. Variables associated with the fringe are denoted with the subscript 0. For example, q_0 is the downstream quantity of the fringe, and β_0 the fringe's retailing cost elasticity. We will only consider the constant elasticity case here, although the more general model follows directly.

The efficient solution satisfies:

$$0 = Q^{-1/\alpha} - \left(\frac{Q-q_0}{K}\right)^{1/\beta} - \left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta},$$

$$\left(\frac{q_0}{k_0}\right)^{\frac{1}{\beta_0}} = \left(\frac{Q-q_0}{K}\right)^{1/\beta} \text{ and } \left(\frac{x_0}{\gamma_0}\right)^{\frac{1}{\eta_0}} = \left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta}$$

Thus

$$\left(\frac{Q-q_0}{K}\right) = \left(\frac{q_0}{k_0}\right)^{\frac{\beta}{\beta}}$$

Therefore,

$$0 = \frac{dQ}{K} - \frac{Q-q_0}{K^2}dK - dq_0\left(\frac{1}{K} + \frac{\beta}{\beta_0}\left(\frac{q_0}{k_0}\right)^{\beta/\beta_0 - 1}\frac{1}{k_0}\right)$$

$$= \frac{dQ}{K} - \frac{Q-q_0}{K^2}dK - dq_0\left(\frac{1}{K} + \frac{\beta}{\beta_0}\frac{1}{q_0}\frac{Q-q_0}{K}\right)$$

$$dq_0 = \frac{dQ - \frac{Q-q_0}{K}dK}{1 + \frac{\beta}{\beta_0}\frac{Q-q_0}{q_0}} = \frac{dQ - \frac{Q-q_0}{K}dK}{1 + \frac{\beta}{\beta_0}\frac{1-s_0}{s_0}}.$$

Similarly,

$$dx_0 = \frac{dQ - \frac{Q-x_0}{\Gamma} d\Gamma}{1 + \frac{\eta}{\eta_0} \frac{1-\sigma_0}{\sigma_0}}.$$

Differentiating the price=marginal cost equation,

$$0 = dQ\left[-\frac{1}{\alpha} Q^{-1/\alpha - 1} - \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta - 1} \frac{1}{K} - \frac{1}{\eta}\left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta - 1} \frac{1}{\Gamma} \right]$$

$$+ \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta - 1} \frac{dq_0}{K} + \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta - 1} \frac{Q-q_0}{K^2} dK + \frac{1}{\eta}\left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta - 1} \frac{dx_0}{\Gamma}$$

$$= \frac{dQ}{Q}\left[-\frac{1}{\alpha} Q^{-1/\alpha} - \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta} \frac{Q}{Q-q_0} - \frac{1}{\eta}\left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta} \frac{Q}{Q-x_0} \right]$$

$$+ \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta} \frac{1}{Q-q_0} \frac{dQ - \frac{Q-q_0}{K} dK}{1 + \frac{\beta}{\beta_0} \frac{1-s_0}{s_0}} + \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta} \frac{dK}{K}$$

$$+ \frac{1}{\eta}\left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta} \frac{1}{Q-x_0} \frac{dQ}{1 + \frac{\eta}{\eta_0} \frac{1-\sigma_0}{\sigma_0}}$$

$$= \frac{dQ}{Q}\left[-\frac{1}{\alpha} Q^{-1/\alpha} - \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta} \frac{Q}{Q-q_0} \left(1 - \frac{1}{1 + \frac{\beta(1-s_0)}{\beta_0 s_0}} \right) \right]$$

$$+ \frac{dQ}{Q}\left[-\frac{1}{\eta}\left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta} \frac{Q}{Q-x_0} \left(1 - \frac{1}{1 + \frac{\eta}{\eta_0} \frac{1-\sigma_0}{\sigma_0}} \right) \right] + \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta} \frac{dK}{K} \left(1 - \frac{1}{1 + \frac{\beta}{\beta_0} \frac{1-s_0}{s_0}} \right)$$

$$= \frac{dQ}{Q}\left[-\frac{1}{\alpha} Q^{-1/\alpha} - \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta} \frac{1}{1-s_0}\left(\frac{\beta(1-s_0)}{\beta_0 s_0 + \beta(1-s_0)}\right)\right]$$

$$+ \frac{dQ}{Q}\left[-\frac{1}{\eta}\left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta} \frac{1}{1-s_0}\left(\frac{\eta(1-\sigma)}{\eta(1-\sigma_0)+\eta_0\sigma_0}\right)\right] + \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta} \frac{dK}{K}\left(\frac{\beta(1-s_0)}{\beta_0 s_0 + \beta(1-s_0)}\right)$$

$$= -\frac{dQ}{Q}\left[Ar + \frac{(1-\theta)r}{\beta(1-s_0)+\beta_0 s_0} + \frac{\theta r}{\eta(1-\sigma_0)+\eta_0\sigma_0}\right] + (1-s_0)\frac{dK}{K}\frac{(1-\theta)r}{\beta(1-s_0)+\beta_0 s_0}$$

Define

$$A = \frac{1}{\alpha}, \quad B = \frac{1-\theta}{\beta(1-s_0)+\beta_0 s_0}, \quad C = \frac{\theta}{\eta(1-\sigma_0)+\eta_0\sigma_0}$$

Then,

$$\frac{K}{Q}\frac{dQ}{dK} = (1-s_0)\frac{B}{A+B+C}.$$

$$\frac{d}{dK}\frac{Q-q_0}{K} = \frac{dQ-dq_0}{K} - \frac{Q-q_0}{K^2}dK = \frac{1}{K}\left[dQ - \frac{dQ - \frac{Q-q_0}{K}dK}{1+\frac{\beta}{\beta_0}\frac{1-s_0}{s_0}} - \frac{Q-q_0}{K}dK\right]$$

$$= \frac{1}{K}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(dQ - \frac{Q-q_0}{K}dK\right) = \frac{dK}{K}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{dQ}{dK} - \frac{Q-q_0}{K}\right)$$

$$= \frac{dK}{K}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{K}{Q}\frac{dQ}{dK} - \frac{Q-q_0}{Q}\right)\frac{Q}{K} = \frac{dK}{K}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{(1-s_0)B}{A+B+C} - (1-s_0)\right)\frac{Q}{K}$$

$$= \frac{dK}{K}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{B}{A+B+C} - 1\right)\frac{Q-q_0}{K} = \frac{dK}{K}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\frac{Q-q_0}{K}$$

$$\frac{d}{dK}\frac{Q-x_0}{\Gamma} = \frac{1}{\Gamma}\left[\frac{dQ}{dK} - \frac{dQ/dK}{1+\frac{\eta}{\eta_0}\frac{1-\sigma_0}{\sigma_0}}\right] = \frac{1}{\Gamma}\frac{\eta(1-\sigma_0)}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{Q-q_0}{K}\frac{B}{A+B+C}$$

The firm's profits are

$$\pi = r(Q)\frac{\hat{k}_i}{K}(Q-q_0) - k_i w\left(\frac{\hat{k}_i}{K}\frac{(Q-q_0)}{k_i}\right) - \gamma_i c\left(\frac{\hat{\gamma}_i}{\Gamma}\frac{Q-x_0}{\gamma_i}\right) + p\left(\frac{\hat{\gamma}_i}{\Gamma}(Q-x_0) - \frac{\hat{k}_i}{K}(Q-q_0)\right)$$

$$= w'\left(\frac{Q-q_0}{K}\right)\frac{\hat{k}_i}{K}(Q-q_0) - k_i w\left(\frac{\hat{k}_i}{K}\frac{Q-q_0}{k_i}\right) + c'\left(\frac{Q-x_0}{\Gamma}\right)\frac{\hat{\gamma}_i}{\Gamma}(Q-x_0) - \gamma_i c\left(\frac{\hat{\gamma}_i}{\Gamma}\frac{Q-x_0}{\gamma_i}\right).$$

$$\frac{\partial \pi}{\partial \hat{k}_i} = (w' - w_i')\frac{\partial}{\partial \hat{k}_i}\frac{\hat{k}_i(Q-q_0)}{K} + w''\left(\frac{d}{dK}\frac{Q-q_0}{K}\right)\frac{\hat{k}_i(Q-q_0)}{K}$$
$$+ (c' - c_i')\frac{\partial}{\partial \hat{k}_i}\frac{\hat{\gamma}_i(Q-x_0)}{\Gamma} + \left(c''\right)\left(\frac{\hat{\gamma}_i}{\Gamma}\frac{d}{dK}(Q-x_0)\right)\frac{\hat{\gamma}_i}{\Gamma}(Q-x_0)$$

$$= (w' - w_i')\left(\frac{(Q-q_0)}{K} - \frac{\hat{k}_i}{K}\left(\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\frac{Q-q_0}{K}\right)\right)$$

$$+ w''\left(\frac{1}{K}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\frac{Q-q_0}{K}\right)\frac{\hat{k}_i(Q-q_0)}{K}$$

$$+ (c' - c_i')\frac{\hat{\gamma}_i}{\Gamma}\frac{\eta(1-\sigma_0)}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{Q-q_0}{K}\frac{B}{A+B+C}$$

$$+ c''\frac{\hat{\gamma}_i}{\Gamma}\frac{\eta(1-\sigma_0)}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{Q-q_0}{K}\frac{B}{A+B+C})\frac{\hat{\gamma}_i}{\Gamma}(Q-x_0)$$

$$= (w' - w_i')\frac{(Q-q_0)}{K}\left(1 - \frac{s_i}{1-s_0}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\right)$$

$$+ \frac{w'}{\beta}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\frac{s_i}{1-s_0}\frac{Q-q_0}{K}$$

$$+ (c' - c_i')\frac{\sigma_i}{1-\sigma_0}\frac{\eta(1-\sigma_0)}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{Q-q_0}{K}\frac{B}{A+B+C}$$

$$+ \frac{c'}{\eta}\frac{\sigma_i}{1-\sigma_0}\frac{\eta(1-\sigma_0)}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{Q-q_0}{K}\frac{B}{A+B+C})$$

Thus,

$$0=(w'-w_i')\left(1-\left[\frac{\beta s_i}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\right)+w'\left[\frac{s_i}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)$$

$$+(c'-c_i')\frac{\eta\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{B}{A+B+C}+c'\frac{\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{B}{A+B+C}$$

Thus,

$$0=\frac{(w'-w_i')}{r}\left(1-\left[\frac{\beta s_i}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\right)+(1-\theta)\left[\frac{s_i}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)$$

$$+\frac{c'-c'}{r}{}_i\frac{\eta\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{B}{A+B+C}+\theta\frac{\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{B}{A+B+C}$$

Employing the definitions $\psi_i=\dfrac{w'-w_i'}{r}$, $\chi_i=\dfrac{c'-c_i'}{r}$ and using the analogous equation from the optimization of $\hat{\gamma}_i$ we have

$$\begin{bmatrix} A+B+C-\dfrac{\beta(A+C)s_i}{\beta(1-s_0)+\beta_0 s_0} & \dfrac{\eta B\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0} \\ \dfrac{\beta C s_i}{\beta(1-s_0)+\beta_0 s_0} & A+B+C-\dfrac{\eta(A+B)\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0} \end{bmatrix}\begin{pmatrix}\psi_i\\ \chi_i\end{pmatrix}=\begin{pmatrix}B(A+C)s_i-BC\sigma_i\\ C(A+B)\sigma_i-BCs_i\end{pmatrix}$$

Let $\overline{\beta}=\beta(1-s_0)+\beta_0 s_0$ and $\overline{\eta}=\eta(1-\sigma_0)+\eta_0\sigma_0$. Then the determinant of the LHS matrix is

$$DET=(A+B+C)^2-(A+B+C)\left[\frac{\beta}{\overline{\beta}}(A+C)s_i+\frac{\eta}{\overline{\eta}}(A+B)\sigma_i\right]+\frac{\beta}{\overline{\beta}}\frac{\eta}{\overline{\eta}}\left[(A+C)(A+B)-BC\right]s_i\sigma_i$$

$$=(A+B+C)\left[(A+B+C)-\left[\frac{\beta}{\overline{\beta}}(A+C)s_i+\frac{\eta}{\overline{\eta}}(A+B)\sigma_i\right]+\frac{\beta}{\overline{\beta}}\frac{\eta}{\overline{\eta}}As_i\sigma_i\right]$$

$$=(A+B+C)\left[A\left(1-\frac{\beta}{\overline{\beta}}s_i\right)\left(1-\frac{\eta}{\overline{\eta}}\sigma_i\right)+B\left(1-\frac{\eta}{\overline{\eta}}\sigma_i\right)+C\left(1-\frac{\beta}{\overline{\beta}}s_i\right)\right]$$

Thus

$$\begin{pmatrix} \psi_i \\ \\ \chi_i \end{pmatrix} = \frac{1}{DET}\begin{bmatrix} A+B+C-\dfrac{\eta(A+B)\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0} & \dfrac{-\eta B\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0} \\ \dfrac{-\beta C s_i}{\beta(1-s_0)+\beta_0 s_0} & A+B+C-\dfrac{\beta(A+C)s_i}{\beta(1-s_0)+\beta_0 s_0} \end{bmatrix}\begin{pmatrix} B(A+C)s_i - BC\sigma_i \\ \\ C(A+B)\sigma_i - BCs_i \end{pmatrix}$$

Thus,

$$\psi_i = \frac{B}{DET}\left[\left(A+B+C-\frac{\eta}{\overline{\eta}}(A+B)\sigma_i\right)\left((A+C)s_i - C\sigma_i\right)-\frac{\eta}{\overline{\eta}}\sigma_i\left(C(A+B)\sigma_i - BCs_i\right)\right]$$

$$= \frac{B}{DET}\left[(A+B+C)\left((A+C)s_i - C\sigma_i\right)-\frac{\eta}{\overline{\eta}}(A+B)\sigma_i\left((A+C)s_i\right)-\frac{\eta}{\overline{\eta}}\sigma_i\left(-BCs_i\right)\right]$$

$$= \frac{B}{DET}\left[(A+B+C)\left((A+C)s_i - C\sigma_i\right)-\frac{\eta}{\overline{\eta}}(A+B+C)As_i\sigma_i\right]$$

$$= \frac{B(A+B+C)}{DET}\left[\left((A+C)s_i - C\sigma_i\right)-A\frac{\eta}{\overline{\eta}}s_i\sigma_i\right] = \frac{B(A+B+C)}{DET}\left[C(s_i-\sigma_i)+As_i\left(1-\frac{\eta}{\overline{\eta}}\sigma_i\right)\right]$$

$$= \frac{B\left[C(s_i-\sigma_i)+As_i\left(1-\frac{\eta}{\overline{\eta}}\sigma_i\right)\right]}{A\left(1-\frac{\beta}{\overline{\beta}}s_i\right)\left(1-\frac{\eta}{\overline{\eta}}\sigma_i\right)+B\left(1-\frac{\eta}{\overline{\eta}}\sigma_i\right)+C\left(1-\frac{\beta}{\overline{\beta}}s_i\right)}$$

Analogously,

$$\chi_i = \frac{C\left[B(\sigma_i-s_i)+A\sigma_i\left(1-\frac{\beta}{\overline{\beta}}s_i\right)\right]}{A\left(1-\frac{\beta}{\overline{\beta}}s_i\right)\left(1-\frac{\eta}{\overline{\eta}}\sigma_i\right)+B\left(1-\frac{\eta}{\overline{\eta}}\sigma_i\right)+C\left(1-\frac{\beta}{\overline{\beta}}s_i\right)}$$

Computations are performed as follows. First, note that

$$\left(\frac{s_i Q}{k_i}\right)^{\frac{1}{\beta}} = w_i' = w' - r\psi_i = r(1-\theta-\psi_i)$$

or,

$$k_i = s_i Q[r(1-\theta-\psi_i)]^{-\beta}$$

Similarly,

$$\gamma_i = \sigma_i Q[r(\theta-\chi_i)]^{-\eta}$$

Choose price and quantity units so that the initial price and quantity are both unity. Then capacities are readily computed from these equations, and are correct up to a scalar. The price is generally $r = Q^{-A}$, and thus the effect of a change in the allocation of the capacities can be computed by solving for the market shares the equations

$$F = k_i^{1/\beta}(1-\theta-\psi_i) - s_i^{1/\beta} Q^{1/\beta + A} = 0$$

and $G = \gamma_i^{1/\eta}(\theta-\chi_i) - \sigma_i^{1/\eta} Q^{1/\eta + A} = 0$

The strategy for computation is to guess values of Q and θ, then see if the firms' desired shares, as given by the solutions to F=G=0, sum to one. F and G have the useful properties that they are both positive at $s_i=\sigma_i=0$, F is increasing in σ_i and decreasing in s_i, and G is decreasing in σ_i and increasing in s_i. Thus, a search starting at (0,0) and always increasing in both arguments finds a solution for the desired shares although the solution may be the upper bound of 1. These solutions provide a sum of shares; the two equations

(*) $\quad \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \sigma_i = 1$

are then used to compute the solution for Q and θ, the remaining unknowns. The unknowns are bounded, θ in [0,1] and Q greater than zero and no greater than the efficient quantity. The latter is calculable directly from the capacities. The desired shares are decreasing in the total quantity Q, guaranteeing a unique solution in Q for any given θ. Generally, as θ rises, the desired values of σ_i rise and s_i fall, but we don't have a proof for this. Mathematica 3.0 invariably finds a solution while Mathematica 2.2. did not.